Low-attaining students' representational strategies: tasks, time, efficiency and economy

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There are many potential ways to represent arithmetical tasks, but students' choices may be limited by beliefs that only certain standardised representations are 'legitimate' in school mathematics. Furthermore, concern for the quantity and speed of 'work done' can override opportunities for meaningful engagement with the content. This paper draws on a sample of the informal representational strategies observed during a microanalytic study of 11-15 year old students with low prior attainment in mathematics. In absence of pressure to provide quick answers, or to obtain them in a prescribed manner, students worked flexibly, participating in arithmetical reasoning, attempting and succeeding in tasks they were previously unable to engage with. The relationships between representational strategies, economy and efficiency are discussed in relation to multiplicative thinking. These have pedagogical implications for the representational expectations placed on students with difficulties in mathematics, particularly in learning support and intervention contexts.

Introduction

The representation of numerical properties, structures and processes is central to mathematics. There are specialised usages of symbols, diagrams, and suchlike, in which all UK schoolchildren are expected to become familiar, and in which they are formally tested; as such, these privileged forms are often designated 'formal', 'standard' or 'conventional' in pedagogical and research literature. A variety of less formal (standard, conventional) representational strategies may also be found on observing students at work, including drawing and the co-opting of physical items such as fingers, pieces of stationery, etc. to temporarily embody quantities and their relations. Students' own visuospatial representations have increasingly been of interest in educational research (e.g. Ainsworth, 2006; Cox, 1999; Deliyianni, Monoyiou, Elia, Georgiou, & Zannettou, 2009; DiSessa, 2004) and were the focus of the larger study from which this paper draws. Nevertheless, the perception of formal symbolic products as "the almost sole desired and valued outcomes of mathematics learning" (Karsenty, Arcavi, & Hadas, 2007, p.159) has historically been widespread, and despite considerable critique (for example, from Ben-Yehuda, Lavy, Linchevski, & Sfard, 2005), still persists amongst many involved in decision-making in mathematics education. Note that while this paper necessarily makes reference to students' strategies as ‘informal’ (etc.), this should not be taken to imply a binary of standardised/formal/conventional/etc. versus nonstandard/informal/unconventional arithmetical representation. In fact, the (in)formality of an arithmetical representation is better considered not only as on a continuum, but as being made up of multiple elements of differing levels of formality (Moschkovich, 2004). For example, observe the mixing of drawn organisational and decorative elements with number symbols, as seen in Figure 3, Figure 4, Figure 9, and Figure 10 below).

There are many theoretical, methodological and pedagogical issues relating to students’ own representational strategies for arithmetical problem-solving. Karsenty et al. (2007) argue that teachers’ appreciation and legitimisation of students’ informal mathematical products is crucial for the developing understanding of low-attaining students. This may happen in Early Years education; however, for secondary-aged students, they are much less likely to be expected, encouraged or appreciated. This depreciation is a result of various factors including (but not limited to) the perceived characteristics and potential both of the various different types of arithmetical representation, and of the students who employ them (Ben-Yehuda et al., 2005).

A major factor in considering representational alternatives is that of time. Decisions on how to make best use of time available (i.e. concerns of *efficiency* and *economy*) are key in school mathematics. These decisions affect all students, but here the focus is those who have struggled for years with numeracy – a highly heterogeneous set of individuals frequently lumped together as ‘low-attaining’ (or equivalent). These ‘left-behind’ children have been in recent years (in the UK) increasingly the recipients of an expansion in in-class support from Teaching Assistants (TAs), and/or separate tuition in small groups or individually. However, there have been concerns raised about the lack of, or even negative, impact of TAs in general (Blatchford, Russell, & Webster, 2012), and specifically in numeracy (Muijs & Reynolds, 2003). One might expect being selected for additional support to be an opportunity for contingent tuition (Broza & Kolikant, 2015) for the individual, at appropriate pace, including taking the time to locate conceptual weaknesses and discover appropriate representational strategies. However, like teachers, TAs often report pressure to ‘cover’ a certain amount of curriculum 'ground' in a given number of minutes or hours. In this paper I argue that this approach is not beneficial, and that to provide meaningful support and encouragement for low-attaining students requires a rethinking of tasks, time, efficiency and economy.

The larger project from which this paper is drawn (Finesilver, 2014) was designed to foreground informal representational strategies, explicitly encouraging students with a history of very low performance in school mathematics assessments compared to their peers (the definition of ‘low-attaining’ used) to experiment with drawn and concrete representations for arithmetical tasks. One of its main methodological features was a radical removal of time constraints, slowing down the pace not just in comparison to regular mathematics lessons, but as much as required by each individual. It became clear that arithmetical and representational strategic choices are tightly bound with considerations of time, and a seemingly simple methodological decision – not to 'rush' participants – was in fact a rather more complex issue worthy of consideration in its own right.

The specific questions addressed in this paper are:

- What are the consequences of removing unnecessary constraints on time and methods allowed, while explicitly encouraging low-attaining students to be creative in their representational strategies for division-based tasks?
- What are the learning and teaching implications regarding *efficient strategies* and *economy of learning* for these students?

Efficiency

The current version of the National Curriculum for England (DfE, 2014) does not specifically mention efficiency at Key Stage 3 or 4 (age 11-16, the group addressed in this paper), but does at Key Stage 2 (age 7-11), which requires that pupils "develop efficient written and mental methods" for calculating, "for example, using commutativity and associativity . . . and multiplication and division facts . . . to derive related facts". Here, by 'efficient', they mean that once a specified collection of arithmetical facts and symbolic algorithms have been memorised, these may be deployed so to minimise the average time taken to achieve an answer for each multiplication or division calculated. For such memory-based methods Krainer (1993) coins the useful metaphor of "motorways", and a great deal of research has concerned itself with the fastest of motorways, and how learners may achieve proficiency on them. While there has been rightful critique of teachers over-emphasising efficiency at the expense of understanding (e.g. Thompson, 2010), there is little questioning of the notion of 'efficiency' itself.

Returning to the National Curriculum, there is conflict regarding the official status of standardised notations: while 'Appendix 1' sets out "some examples of formal written methods for all four operations to illustrate the range of methods that could be taught", and states that these are "not intended to be an exhaustive list", the national KS2 tests allocate 'method marks' for multidigit multiplication and division tasks such that students not using the prescribed layout for their calculation may be penalised. While formal testing is not my focus, this highlights some general assumptions made about efficient work in school mathematics (especially but not restricted to arithmetic; there are equivalents in algebra and other topics). These include:

- Efficiency refers to how quickly a given type of calculation (or solution) can be performed.
- Calculation (or solution) methods can be compared such that some are deemed absolutely more efficient than others.

One of the definitions of ‘efficiency’ given in the Oxford English Dictionary is “Fitness or power to accomplish, or success in accomplishing, the purpose intended; adequate power, effectiveness, efficacy”. This prompts the question of the actual intended purpose of requiring students to perform arithmetical calculations. Historically it has been valuable to reckon accurately in the absence of computers, calculators, abacuses, or other assistive technologies, and thus standard methods have had a secure place in the mathematics curriculum. Arguments for and against the continuation of this are legion; however, let us assume for now that the ability merely to replicate a series of (seemingly) arbitrary steps is not the intended purpose. Perhaps it is also to make strategic use of the decimal place value system, arithmetical relationships and principles, and logical reasoning to work out unknown information from known information? In this case, if arithmetical activity outpaces conceptual understanding, as happens with many students (such as those discussed below), the intended purpose is lost, and so any claim to efficiency.

I suggest that while it is not impossible to compare two or more calculation methods for their efficiency, it cannot be done independently of the numbers involved, type of task in which they are embedded, the individual carrying them out, and their current circumstances. E.g. for certain number combinations, the formal long division algorithm is neither the quickest nor easiest (like the motorway for certain journeys, in Krainer’s metaphor).

> The fact that people use diverse strategies is not a mere idiosyncrasy … Strategies differ in their accuracy, in the amounts of time needed for execution, in their memory demands . . . Wise choices of strategies allow people to meet situational demands and to overcome limited knowledge and processing resources. (Siegler, 1988)

So, one might ask what ‘efficient’ calculation means for individuals who, for whatever reason, do not (yet) have a convenient and reliable mental bank of facts and procedures.

Economy

The term ‘economy’ is sometimes used synonymously with ‘efficiency’, e.g. “reorganization of work tasks to reduce the number of physical or mental steps required for their accomplishment and/or to simplify steps that cannot be eliminated” (Scribner, 1997). However, like Gattegno and others, I reserve it for a longer view of educational practices – as a function of curriculum over time rather than task over time.

> [T]ime, which for every individual is the stuff of life, is not considered by teachers as having any value. Teachers are prepared to repeat and repeat, review and review, correct and correct, as many times as they face a given group of students. (Gattegno, 1971, p.73)

> The economics of education as such is quite simple … students' time must buy equivalent experience. (ibid, p.75)

I would argue that, over 40 years on, it is no longer true (if it ever were) to claim that teachers do not value time – indeed, time pressure is a major source of professional stress – yet this critique of pedagogical practice is still relevant, and particularly so for low-attaining students. This is in consequence of further general assumptions, that:

- All can learn to retrieve the required facts and reproduce the required methods, although for some individuals it will take more time and effort.
- Until they do, this is best use of their time.

On spending considerable time in the UK secondary education system (as a teacher, researcher, and staff training provider), it is clear to me that many teachers, support staff, and others (including parents and Ministers for Education) consider those mathematical activities that produce pages of easily-recognisable 'work' such as rows of tidy sums to be unequivocally good use of time, and those that do not, to be time wasted. I am not proposing an end to either practising written methods or memorisation of useful information, but am concerned for those students who, despite many years of hours and hours of mathematics lessons, and in many cases, additional numeracy support, show the kind of stubborn lack of improvement that indicates a pedagogical issue. Here, prescribing more of the same approach, while expecting a different outcome, is likely to be time wasted.

So, one might ask what 'economical' learning means for an adolescent whose arithmetical understanding is (as yet) partial and/or insecure.

Expectations

Is the mathematical education of those students with prior low attainment, in fact, intended to be economical? This does not seem the case, when their lessons deal in "simplified mathematics, broken down into step-by-step processes, offered in short chunks … a fragmented, mechanistic approach", with students "praised for trivial performance" (Watson, 2006, p.103) yet still complaining that they "do not understand a topic after they have met it over and over again" (Ahmed, 1987, p.14). If attainment is measured by performance of the steps resulting from a chain of reasoning in which the learner was not involved, then in "getting to the end of the work … filling in the required partial answers", students are implicitly taught that they should not expect more, and that "engagement with expected classroom behaviour and observable completion of clerical work" (Watson, p.104) equates to successful school mathematics. When these students describe the work as an irrelevant, boring waste of time (e.g. Cooper & McIntyre, 1996; Boaler, 1997), it is worth considering their point.

The prescription, according to these authors, is increased challenge – within a supportive environment. There, the students "*can* cope with the frustrations and floundering inherent" (Ahmed, 1987, p.14) and "think in ways normally attributed to successful mathematicians" (Watson, 2006, p.116). There is a counter-argument sometimes made, namely that the more calculations completed, the more data students will have from which to engage in pattern recognition (universally agreed as central to mathematical thinking), and come to understand the underlying arithmetical concepts. Unfortunately, in normalised classroom practice where quantity of visible 'work done' is prized, taking precious time out from its production to think more deeply about it will not be prioritised. (This is exemplified by textbook exercises which consist of half a page of calculations, followed by an optimistic but generally ignored "What do you notice?".)

Counting and multiplicative thinking

The examples discussed in this paper focus specifically on first developments in multiplicative thinking. There is known to be considerable variation in the ease and manner in which individual learners develop the many aspects and levels involved (Brown et al, 2010; Mabbott & Bisanz, 2008; Zhang et al, 2013). The use of counting as a main strategy in multiplicative-structured tasks is generally associated with primary-age children, but persists much later as a backup strategy. There is an important distinction between retaining counting as a backup, and relying on it as sole strategy (Dowker, 2005), as is more likely for children with arithmetical difficulties (Geary et al, 1992; Gray & Tall, 1994; Ostad, 1997; Siegler, 1988). However, grouped and rhythmic unitary counting reinforces understanding of the structure of the problem, giving them a model on which to base later abstraction (Anghileri, 1997). Students' concrete representations of equal-groups arithmetical relationships have been analysed in detail by Anghileri (1989), Kouba (1989), and Mulligan & Mitchelmore (1997) for the information they provide on learners' developing understandings of natural-number operations. Increasingly efficient counting strategies were observed, such as grouped and step-counting, that led to repeated addition and eventually recalled multiplication facts.

However, strategic development in arithmetic consists not of the replacement of a single immature strategy by a single more mature strategy but of the discovery of increasingly more mature strategies, which co-exist for a long time with immature strategies (Dowker, 2005; Voutsina, 2012) Change is not always as might be expected, and children's self-reporting can give insight into puzzling findings, such as when Baroody et al (1983, in Baroody & Ginsburg, 1986, p.102) found an efficient arithmetical 'shortcut' being used far more frequently by younger children than older ones (with similar results from Bisanz et al (1984), ibid). One girl's comment that "I cheated on that one; I looked at the [previous calculation]" led to the plausible explanation that after greater exposure to school mathematics, children might come to believe that when set an arithmetic problem, they are supposed to calculate it 'properly' (i.e. in the standard method their teacher had demonstrated), and that making efficient use of any helpful patterns and principles to shorten the task was tantamount to cheating! (Baroody & Ginsburg, 1986)

Visuospatial representation

"Draw a figure" is one of the first suggestions in Pólya's 1945 classic 'How to solve it', and it is notable that the suggestion appears under the first of his four stages of problem solving, 'Understanding the problem', as opposed to later on in the process. However, experienced teachers know that simply telling students to draw will not do: while some may appear to be natural 'visualisers' (in fact, too simplistic and one-dimensional a characterisation (Cox, 1999; Kozhevnikov, Kosslyn, & Shephard, 2005)), many others ask "Draw what?". Mason & Davis (1991) discuss those students who would likely benefit from drawing in problem-solving yet do not attempt it, asserting that "Many pupils have no idea where diagrams come from" (p.35), and "many pupils are unaware that the story is intended to evoke mental images, a sense of 'being in the situation' … that enables you to read relationships and operations that need to be carried out" (p.34) – meaning that they do not realise the (usually unspoken) considerations and decisions that must take place to translate tasks into a symbolic or visuospatial format they might use for solution. This awareness, then, must be prompted, but in a way that is individually meaningful to the students, and not simply as another set of steps to copy.

From a constructivist viewpoint, the distinction between reasoning with one's own representations versus those created by others is vital (e.g. Papert, 1993). Some educators have proposed teaching of generalised heuristics and principles for choosing representations for the requirements of the task (Cox, 1999), while others have advocated a more open-ended educational approach, via reports of children creating their own, original and often highly effective, representational forms (examples in DiSessa, 2002, 2004). DiSessa defines the term *metarepresentational competence* (MRC) as a set of abilities for dealing with representational issues, in particular the abilities to create one's own representations for a given purpose, and to critique their adequacy and suitability for that or other purposes. It is notable that many of those who advocate more open, hands-off, creative approaches, and/or teaching MRC, did their research with average-to-high-attaining primary-age students. However, in a rare longitudinal study of secondary-age students, Karsenty et al (2007) recommend wider encouragement of informal, nonstandard representations, concluding that “rushing into formal mathematical outcomes, without taking into consideration the intuitions and informal ideas of students, might weaken potential strengths of … low achievers” (p.175)

Methodology

This project from which this data derives was a qualitative study focusing on the arithmetical-representational strategies employed for multiplication- and division-based scenario tasks. Multimodal data was collected in individual and paired task-based interviews carried out by the author, with a multiple case-study design and microgenetic analysis. *Microgenetic* methods (originally developed for studying the transition processes of cognitive development (Siegler & Crowley, 1991)) are characterised by high-density observation over periods of changing competence, followed by intensive analysis. They have proved appropriate for case studies of individuals with difficulties in mathematics (Fletcher et al, 1998; Schoenfeld et al, 1993), and been used increasingly in studies of children's arithmetical strategies (e.g. Robinson & Dubé, 2008, 2009; Voutsina, 2012). A new analytical framework was developed and deployed during the project for qualitative microanalysis of the multimodal data along multiple aspects of arithmetical-representational strategy (described in Finesilver (in press)).

Participants

The thirteen participants were aged 11-15, attending mainstream Inner London secondary schools, and identified by their teachers and prior educational records as particularly low-attaining in mathematics compared to their peers. A range of Special Educational Needs, disabilities (as defined under the *2010 Equality Act*) and neurodivergences had previously been diagnosed, and their educational profiles, while not typical for the general population, were representative of the diversity found in any non-selective school. Classroom observation and initial sifting assessments had indicated that, whatever the individual etiologies of their difficulties, these young people were all at a very particular stage of arithmetical development: that where one is comfortable with the principles of addition and subtraction but struggling with the principles of multiplication and division. I emphasize *principles* because these individuals made frequent errors and could not necessarily perform the operations outside of low natural numbers; however, their understanding of the operations as metaphorically equivalent to combining and separating sets of objects was sound.

Interviews

Participants each had six 45-minute interviews in total: an initial assessment (near the start of the school year); a series of problem-solving interviews focusing on different aspects of multiplicative thinking with natural numbers; and a follow-up session (near the end of the school year). From a practitioner perspective, the interviews involved checking the integrity of the conceptual foundations of multiplicative relationships for each individual, and fixing weak, incomplete or missing links, to enable progress towards a more solid, understanding-based use of those mathematical symbols with which by this stage of schooling all were familiar, yet far from comfortable. From a research perspective, the tasks were designed to illuminate some of the unseen difficulties low-attaining students have with multiplication and division, and in the progression from additive to multiplicative thinking, via their representations.

Tasks

A full list of tasks may be found in (Finesilver, 2014), and detailed analysis of two particular tasks in (Finesilver, in press, 2009). The two main scenarios employed were 'Biscuits' (partitive division, where a number of biscuits are to be shared between a number of children) and 'Passengers' (quotitive division, where the number of vehicles required to transport a given number of passengers is calculated), and also bare multiplication and division calculations (i.e. presented without scenario). The quantities involved were selected flexibly in situ, based on the capabilities of the individual as observed. The available representational media were multilink cubes, coloured pens and paper.

Two overall methodological principles were key: encouragement of students' own representational ideas, needs and preferences, and complete absence of time pressure on tasks.

Findings

Maths-like behaviour

The participants in this study were willing students who generally attempted to work hard and do what they thought I wanted of them. Note the *'thought'*: initial responses to tasks involved high levels of what I term *maths-like behaviour* (Finesilver, 2014), i.e. engaging one's efforts in activities which give the superficial appearance of 'doing maths' but do not actually involve much, if any, mathematical thinking.

Two main kinds of maths-like behaviour were observed, the first being the quick assertion of (often unrelated) numbers for answers, with no attempt to calculate, despite having the means to do so. To illustrate, when I asked one twelve-year-old how he had obtained his incorrect answer to a multiplication, he explained he had picked one of his “lucky numbers” (something he claimed also to do in examinations!). Having seen their peers call out ‘times tables’ facts on demand in class, with no visible calculation, as if pulling them by magic from thin air, they had developed the imitative behaviour required for ritual participation (Sfard, e.g. 2016) in this familiar classroom ceremony. These guesses were frequently wrong, but the rarer instances when a number bond was correctly recalled appeared to reinforce their belief in it as an appropriate way to respond to arithmetical tasks. I assert that, as suggested by Watson (above) this had been further reinforced by the delighted approval adults tend to give to low-attaining students’ correct answers, regardless of provenance. This urge to give this type of encouragement is understandable, but creates a lottery-like atmosphere, where one picks a number and is told by the teacher if it is good or bad – this is essentially unmathematical and must be avoided.

The second form of maths-like behaviour also relates to relying on memory of things that have not actually been memorised. Students sometimes produced written calculations that imitated the general appearance of standard notational forms, as illustrated in Figure 1-Figure 2 (both by twelve-year-olds). The intention in such cases seemed to be a genuine attempt to ‘please teacher’ by not only providing an Answer, but participating in the activity ‘Showing Working’ (i.e. writing the numbers provided in particular configurations, with additional marks and symbols).

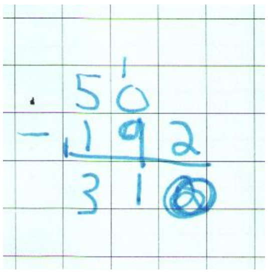

Figure 1: 192÷50

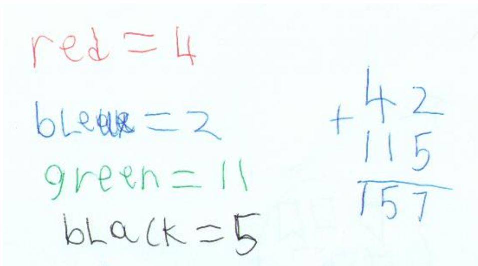

Figure 2: 4+2+11+5 (stage within longer task)

Figure 1 represents a Passengers task: with 192 people traveling in 50-seater coaches, finding the number of coaches needed. The student’s first attempt produced the figure of 310 coaches, which she presented to me as her answer, secure in the knowledge that she had arranged some symbols in a manner that looked ‘mathematical’, with a vague hope that I might pronounce the end product correct, and no expectation of working through a challenge, learning anything new, or even becoming more proficient in a procedure. On inspecting any of the many examples such as these, there are clearly fragments of notation and algorithm which have been retained, but they are surrounded by nonsensical activity. As before, I suggest these behaviours had previously been reinforced by praise for low-attaining students’ efforts in making maths-like marks on the page, and correction only of isolated misremembered elements, without recourse to arithmetical reasoning.

On being told their initial solutions to tasks were incorrect, individuals had different responses. Some repeated their previous actions, i.e. they were motivated to obtain a correct solution and believed in the efficacy of their strategy, but mistrusted their ability to have carried it out properly. Some accepted the failure and simply waited for the next instruction, with little interest in either the true solution or why their strategy had not worked. Others were engaged enough to argue with me and insist their answer as correct. However, none independently responded by thinking critically about the strategy they had used and improving it or attempting an alternative. Additionally, while none were surprised to be wrong, they expressed surprise when I would still ask how they had obtained their incorrect answer.

Mathematical behaviour

Despite clearly and repeatedly stating my main interest as being "how you go about working it out", "your thinking", and "different ways of doing it", it was difficult to convince participants that their maths-like behaviours were unwelcome. Nevertheless, as the interviews progressed, all thirteen participants, sooner or later, did adjust to the new expectations. In an environment where they were consistently encouraged to use whichever forms and elements they felt best suited their needs, and were never hurried but knew they would be asked to justify their solutions, they began to make representational choices that involved genuine (if basic) arithmetical reasoning. By the end, all participants were attempting to engage thoughtfully with tasks via meaningful representations at least some of the time, and some participants did so most or all of the time. They increasingly prioritised working toward an answer in which they had confidence, via a strategy they understood all parts of.

For many, this secure working required a return to *unitary* representations (one mark representing one item), in which the equal-groups structure of multiplication and division (with natural numbers) was clearly visible (as in Figure 3Figure 8). A wide variety of visuospatial representations were created, including imagery and concrete models mixed with numeric symbols (only a few of which may be included here; see Finesilver (2014)for the full set). Particularly popular were *container* and *array* forms (see Figure 3 and Figure 4, respectively).

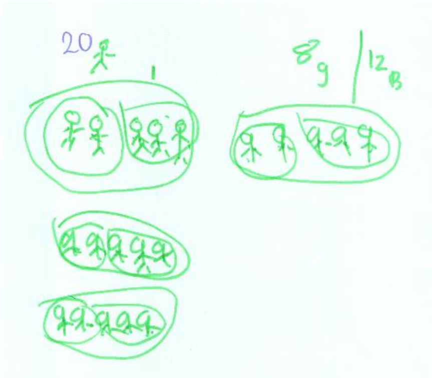

Figure 3: Five groups, each including 2 girls and 3 boys

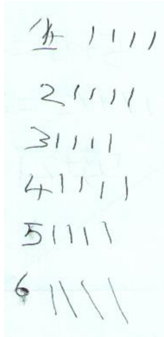

Figure 4: 24 biscuits shared between 4 people

Some students chose very long-winded strategies such as those in Figure 5, which nevertheless empowered them to work through tasks in ways that were both successful and meaningful (which attempting to retrieve number facts and manipulate arrangements of symbols was not), and to succeed on tasks they had otherwise been unable to complete.

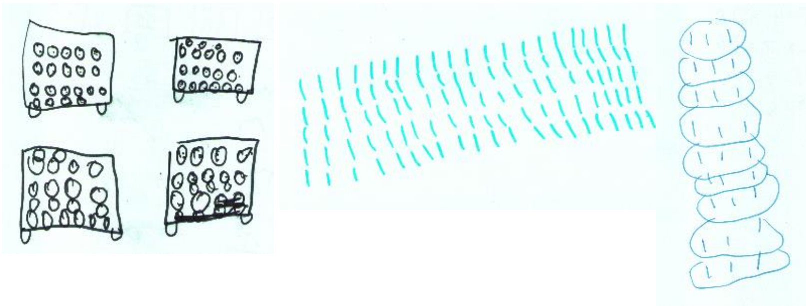

Figure 5: Different types of unitary representation

When some students found a representation type that helped for one task type, they transferred it for use with other task types. For example, the unit containers in Figure 6 could equally represent 30 biscuits shared between 5 people, 30 people grouped in 6-seater taxis, 6 multiplied by 5, etc., reinforcing the underlying equal-groups structure that leads to comprehension of the commutative principle of multiplication, and the inverse relationship between multiplication and division.

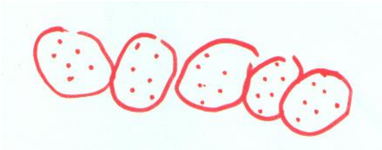

Figure 6: Relationship between 5, 6 and 30

Independent exploration of such arithmetical principles and relationships was a motivating factor in some cases, for which the unconstrained task time was particularly relevant. For example, after completing a task involving enumerating a 3D array of cubes (see Finesilver (in press)), two participants expressed the desire to redo the task with the cuboid in a different orientation, to 'prove' to themselves that the total was still the same. Similarly, after completing a requested grouping of a 2D dot array, creating an *array-container blend*, one fourteen-year-old participant spontaneously worked through all the different possible equal groupings (Figure 7). These students would certainly have been taught the relevant principles at some previous point, but their excitement in independent discovery was striking. This occurred due to generosity of time – which included allowing students to decide that a task was not 'over' yet, and extend it in a way that intrigued them.

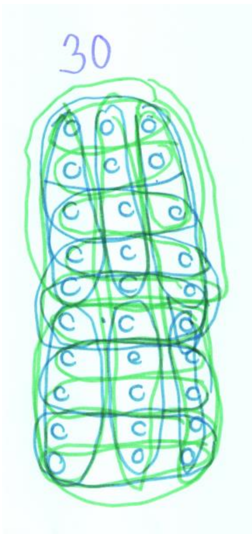

Figure 7: Factors of 30

Close observation of participant representations over a period of time and multiple tasks allowed tracking of strategic change, and so demonstration of a level of judgement not generally assumed to be present in those categorised as low-attaining. For example, there were instances when participants effectively judged which representational elements were helpful to their thinking, discarding those that were not, or no longer, necessary. Those using unit-based representations decided whether and when to drop organisational or decorative elements (Figure 8), and when to make the move from unitary representations to, e.g., *number containers* (Figure 9). In other words, they exhibited *metarepresentational competence* (DiSessa, 2004), and in general the evidence supports the idea of students with arithmetical difficulties being able to make strategic representational choices that allow them to work on tasks in a way which is the most efficient for them *at that point*.

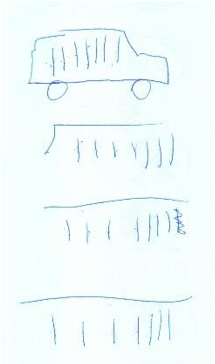

Figure 8: Decreasing decorative elements

Figure 9: Number containers (retaining decorative elements)

Although individual student trajectories varied, the overall direction of progress could be described as a gradual change of focus from units to groups. Whether the tasks were multiplication or division-based (e.g. Biscuits, Passengers), there is the arithmetical structure of a total quantity which is made up of, or can be separated into, equal groups; the most basic enumeration strategies involve counting with little or no awareness of the replicatory structure, while the more advanced ones make use of it to enumerate more efficiently. Representationally, while a decrease in non-mathematically functional elements (e.g. wheels on a bus, arms and legs on people) prompted a minor increase in efficiency, it was moving from a need for one mark to stand for one thing (unitary representation), to being confident using one mark to stand for many, that signified the major step change for these students' arithmetical competence.

There was also evidence supporting Hughes' (1991) finding (with younger children), that learners may be comfortable using symbols representing quantities before symbols representing operations. Consider Figure 10, where a thirteen-year-old student independently introduced an addition sign to a containers-based representation (his preferred form), indicating understanding that his strategy for the division task set (100 people in 25-seater buses) was equivalent to repeated addition of 25 up to a total of 100. Here, retention of some pictorial/iconic representational elements allowed for bridging of the gap between informal and formal notation.

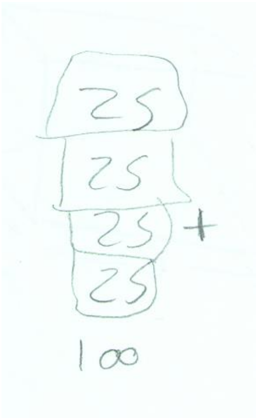

Figure 10: Sum in number containers

These few examples have been selected from a large bank in which they were not atypical, chosen to illustrate some of the ways these low-attaining students made sense of various multiplicative-structured tasks, and some of the small steps by which they progressed independently when ready.

Discussion

I suggest that it is normal for people working on mathematical tasks at all levels to want to minimise their expenditure of time and effort; i.e. to use the most efficient strategies available to them to achieve their goal. When considering nonstandard visuospatial representation of arithmetical tasks (i.e. partial or no use of standard mathematical notation), an efficient strategy:

- includes all the elements which enable the student to solve the task (correctly) more quickly and/or with less effort than they would be able to do without those elements;
- does not include any elements which do not help the student to solve the task more quickly and/or with less effort.

This means that all kinds of imagery may be part of an efficient representation for a given individual at a given time; this includes decorative imagery, which while not mathematically functional, may still serve a valid task-related purpose that accelerates successful solution – for example, to serve as a reminder of some relevant aspects of the task scenario, and so ‘anchor’ students.

However, this does not mean all students *do* actually work in the most efficient way, as this tendency may be modified by their beliefs about desirable and undesirable strategies. Some students, unaware of teachers’ implicitly- or explicitly-expressed expectations of progress, use (e.g.) counting-based strategies that they perceive to be most efficient (for them, on that task, at that time), and in many cases they are correct to do so. Other students, for whom number fact retrieval and compact symbolic notation have repeatedly proved unreliable, still attempt those methods until explicitly encouraged to do otherwise because (a) it is quick and easy to demonstrate their willingness to participate (b) they have seen peers using them and do not wish to appear different, (c) this outward behaviour has been rewarded and reinforced by adults, (d) they believe that alternative representations would incur disapproval (for being too slow, or otherwise unacceptable), or (e) they do not have the metarepresentational competence to create their own reliable representations, and guessing at answers or half-remembered procedures is their only option.

Furthermore, as alluded above, there is a serious problem with low-attaining students not even expecting to understand what they are doing in mathematics lessons, or considering it of any importance to do so. Consider the following casual conversation occurring before an interview:

> CF: What are you up to in maths at the moment?
> Student: Angles.
> CF: How’s that going?
> S: Alright!
> CF: Good!
> S: I don’t really get it though…
> CF: Is there anything in particular you’re finding hard?
> S: Er, angles.

Students may consider their educational experience to be ‘alright’ even while aware they ‘don’t get it’, and they cannot work towards understanding the actions they are instructed to perform if they are unfamiliar with genuine numerical reasoning and unaware that this is an intended goal.

However, these beliefs and behaviours can be shifted significantly to more meaningful, conceptually-based ways of working, at least while they are outside the mainstream classroom. (While the participants in this study were not observed in class subsequent to the interviews, many of the problem-solving strategies chosen in their final interview, at the end of the school year, were similar to those used in the previous interviews, indicating that regardless of their activities in regular lessons, the relevant experiences had not been forgotten in the intervening months.)

It is also worth noting that even in the paired tuition condition, in the presence of just one other student of comparable ability, and despite my efforts to release them from time constraints, there was a small but ineradicable competitive element: one student indicating (in quite neutral manner) that they had finished a task put pressure on the other to complete more quickly. This time-based peer-pressure can be expected to increase in groups of >2, and is not confined to work on closed tasks; while mixed-ability investigational work has been shown as advantageous for some low-attaining students (e.g. by Boaler, 1997), there is a strong possibility of the quicker students' chains of reasoning and exemplification leaving behind those who struggle most with mathematical thinking.

Observing these students charting their own courses, at their own paces, through division tasks that would be straightforward for many but were highly challenging for them, made visible the complexity within a simple-seeming item on the mathematics curriculum. This included complex individual patterns of capability and limitation in developing organisational structures, familiarity with and use of number relationships, and reasoning to derive unknown information from known. It also highlighted representational needs and desires that students are unlikely to express in the classroom. Allowing them to take their time and set the pace functioned in a similar way to allowing them freedom to choose their representational strategies; these are linked, as the lack of time constraint allowed them to choose more time-consuming representations, should these be helpful to their problem-solving. I believe this kind of choice to be not only important for the development of individuals' mathematical thinking, but also empowering for the mathematically disadvantaged.

Conclusions

The set of students in this study initially exhibited symptoms of the kind of educational experiences described previously by Ahmed, Watson, Boaler and others, performing various maths-like behaviours with little expectation of understanding the seemingly-arbitrary rules governing the dance of symbols on page. Similarly, they also responded well to being challenged to reason mathematically. While participants were initially surprised not to immediately receive praise simply for producing 'answers', and in some cases initially unwilling to modify their usual patterns of response, all were at some point later observed engaging in genuine mathematical thinking when the work was set at an appropriate level and under the right circumstances. As discussed above, this did not mean a teacher either breaking tasks up into dissociated chunks, or demonstrating a procedure to be copied. It did mean choosing tasks types and numbers carefully – but not only that. For meaningful progress, these individuals first needed to represent numerical relationships in ways that made complete sense to them at that time, however cumbersome or time-consuming. As Anghileri (2001) found with younger learners, "For developing efficiency, such interpretations cannot be ignored as they represent the pupils' thinking in a way that more formal methods do not" (p.18). Where there was successful movement from basic forms of unitary representation toward symbolic notation, it took place via a path of small and well-connected steps, or microprogressions, at a generally learner-led pace and trajectory, with never too great a cognitive leap between one and the next.

A pedagogical implication is that these students may need more exploration time than currently generally provided. But how much? Progress is not a direct function of time, and it is not possible to predict after how many minutes (or how many similar tasks) either minor adjustments or major step changes will arrive in a given case. However, to err on the side of generosity of time may be considered economical if that time is spent engaging in genuine mathematical challenge.

Regarding the experience of multiple representational forms, it must be remembered that more time-consuming, older strategies in children's problem-solving often persist alongside more advanced alternatives (Dowker, 2005; Voutsina, 2012) rather than being simply replaced. For learners with difficulties in mathematics, such as those discussed here, an apt analogy for strategic multiplicity is with the mobility of a person with some physical illness or injury. The expected progression might be from a pair of crutches to a cane, before – maybe – walking freely. However, the progression is not linear. On one day, the person may be keen to be rid of the supports, and to see if it is possible to manage without; on another, they may feel weaker and retain them. On lacking confidence for a particular trip, it may be appropriate to set out walking 'normally', but with a cane tucked away, ready to be deployed if movement becomes difficult. So it is in arithmetic, with the range of nonstandard visuospatial representational strategies which can act as optional supports for thinking and problem-solving: likewise, a student may on a 'good day' manage without the cognitive support of, say, container representations, but on another, be very glad to have it in their arsenal as a backup.

I have indicated the methodological and diagnostic importance of a general non-interference stance, i.e. stepping back and letting students make their own discoveries at their own pace, including setting their own extension tasks and following trains of mathematical thinking where they lead. However, I have also mentioned encouragement and discussion, i.e. specific moments of 'teacher' input. Where a student indicates that they are 'stuck', the teacher should not address multiple steps at once, but provide the smallest incremental 'nudge' which will allow the student's own thinking process to continue. Meanwhile, taking the time, whenever possible, to invite the student to articulate their working, can give access to tiny steps forward in their thinking (microprogressions) that might not otherwise be seen.

To conclude, Mathematics is not a disconnected mass of arbitrary rules and right/wrong judgements from teachers, but this may well be the main prior experience of low-attaining students. It is, nevertheless, possible to build structured, connected understanding of arithmetic, and while creating the necessary representations may seem to take a long time, it is not wasted time. For all students, but these in particular, the efficiency of different potential strategies should be considered in terms of the context of the individual learner, and economical use of time in terms of quality of mathematical behaviour rather than quantity of maths-like behaviour. Mathematicians of high ability are described as “knowing to fool around with examples” (Watson, 2001, p. 464); this does not have to be their preserve alone, as the benefit can apply to all.

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