Between counting and multiplication: low-attaining students' spatial structuring, enumeration and errors in concretely-presented 3D array tasks

Carla Finesilver, King's College London

Abstract

The move from additive thinking to multiplicative thinking requires significant change in children's comprehension and manipulation of number relationships. This change is not a single cognitive leap, but involves various different conceptual components, and can be a slow, multi-stage process for some learners. Unit arrays are a key visuospatial representation for teaching/learning about multiplicative relationships, but most research focuses on 2D (rectangular) arrays, and that which does address 3D (cuboid) arrays still frequently uses 2D representations of these. This paper documents low-attaining children's partially-developed multiplicative thinking as they work on concretely-presented 3D array tasks; it also presents a framework for microanalysis of learners' early multiplicative thinking in array tasks. Data derives from a small but cognitively diverse set of participants, arithmetically low-attaining for their ages, and relying heavily on counting-based strategies: this enabled detailed analysis of small but significant differences in their arithmetical engagement with the arrays. The analytical framework combines and builds on previous structural and enumerative categorizations, and is appropriate for use with a variety of array representations including but not limited to the blocks of unit cubes in the task sequence described here.

Keywords: numeracy, counting, arithmetical strategies, multiplicative thinking, low attainment, qualitative analysis, microgenetic methods

Between counting and multiplication: low-attaining students' spatial structuring, enumeration and errors in concretely-presented 3D array tasks

This article is concerned with children's conceptualization of 3-dimensional cube arrays as ordered space-filling objects, their interactions with these concrete structures and with the multiplicative relationships they instantiate. Consider the image in Fig. 1, for example, accompanied by a question such as "How many unit cubes make up this block?", as might be seen in many a school mathematics textbook.

[INSERT Figure 1: Perspective drawing of cuboid]

To answer this, a student might determine the three edge lengths of the pictured cuboid and multiply them, that is, use the volume formula; alternatively, they might consider one 'layer' or 'slice' of unit cubes first, then count how many of those layers or slices there were. However, there is a cognitive process that must take place prior to any such calculation: the conceptualization of a quantifiable physical space filled by units ordered in a replicatory pattern, such that they may be enumerated systematically. This is *spatial structuring*, a term originally defined by Battista & Clements (1995) as "the mental act of constructing an organization or form for an object or set of objects". Prior research (e.g. Battista, 1999; Battista & Clements, 1996; Ben-Haim, Lappan, & Houang, 1985) has found that some students have significant difficulties in enumerating the total unit cubes required to construct the object depicted in images such as Fig.1, and, moreover, in conceptualizing these images as ordered, space-filling objects. Additionally, there are certain complicating factors involved for research aimed at delineating the nature, degree, and pedagogical implications of these difficulties.

Firstly, the great majority of enumeration tasks in prior research, like the school texts, have used two-dimensional (2D) drawings of cubes and cuboids, rather than the actual three-dimensional (3D) objects being considered; this mental translation from 2D configurations of lines on a page to an imagined object in 3D space is a major additional cause of difficulty for struggling students. This is a methodological issue, which may be addressed by studying students' interactions with actual concrete cubes and cuboids. Secondly, the development of spatial structuring ability does not happen in isolation. In the case of arrays, it may be developing alongside the ability to consider and reason about multiplicative relationships (frequently termed 'multiplicative thinking'), with the possibility of a two-way relationship between the two. This is an analytical issue, which may be addressed by considering spatial structuring and enumeration strategies as two distinct but related analytical aspects. Thirdly, the kinds of errors students make on this type of task are relevant to analysis of both the spatial and enumerative aspects of their developing multiplicative thinking, and also to pedagogy, in terms of how teacher prompts may most effectively be deployed to support their learning. This may be addressed by treating errors as a separate analytical aspect, and carefully controlling teacher interactions (where such exist).

Why is it important to understand and address children's difficulties interacting with 3-dimensional multiplicative-structured objects in particular? It is educationally relevant not only in terms of conceptualizing and quantifying actual 3-dimensional space, but also in developing the mature multiplicative reasoning skills that play a "pivotal role in the [11-14] curriculum" (Brown, Küchemann, & Hodgen, 2010). Watson (2006) describes the shift from additive to multiplicative thinking as "crucial in secondary mathematics" and identifies it as a particular focus for arithmetically low-attaining students. All learners should have the opportunity to engage in varied multiplicative thinking activities designed to enable them to see pattern and structure in such a way as to move beyond additive and "uni-structural thinking" (Mulligan, 2011). To consider only 2D multiplicative relationships is a significant limitation, whereas the use of 3D structures can extend thinking in even the simplest of tasks (such as those described below).

Research questions

(1) How do arithmetically low-attaining children engage with spatial structure and enumerate in concretely-presented 3D array tasks?

(2) How do their spatial structuring and enumeration strategies relate to their developing multiplicative thinking?

An additional aim of this paper is to demonstrate the application of an enhanced framework developed for the microanalysis of learners' multiplicative thinking in array tasks.

For these purposes it is necessary to review the role of visuospatial representations in general for supporting multiplicative thinking (in particular for students with a history of low attainment in mathematics), the specific functions of the array as a representation of simple multiplicative structure, and the ways these students count, or otherwise enumerate, in multiplicative tasks. An analytical framework may then be constructed encompassing all these aspects.

Visuospatial representations for supporting multiplicative thinking

While the computational aspect of multiplication of natural numbers is often first introduced to students as repeated addition (Clark & Kamii, 1996; Dowker, 2005; Izsák, 2005), the static, structural aspect of multiplication is also usually part of their experience, concretely represented as a set of equal sets of objects, or less frequently, a regular structure made up of a set of equally-sized parts. This cognitive mechanism of reasoning about one thing (objects) as if it were another (numbers), known as *conceptual metaphor* (Lakoff & Núñez, 2000), has historically been a widely-used means of supporting learners' engagement with numbers and arithmetical relationships, specifically through the conceptual metaphors 'Arithmetic as object collection' and 'Arithmetic as object construction' (ibid.). When interacting with these discrete visuospatial representations of multiplicative relationships (e.g. physically-present or pictured items spatially organized into countable equal sets), children initially enumerate uni-structurally, i.e. by individually counting each directly-perceptible unit. At a later stage of education, they are expected to use recalled or derived multiplication facts to perform a purely symbolic operation involving the number in each group and the number of groups. Between these two stages there is significant arithmetical middle ground, which is more developmentally complex than simply identifying multiplication as calculable by repeated addition and/or rote-learning multiplication facts.

The pace at which learners make this development varies, and some children become so skilled in repeated addition and deriving number facts that it is difficult to tell the difference between calculation strategies and direct retrieval (Sherin & Fuson, 2005). For other learners one or both of these arithmetical and representational developments are slow, effortful and occur in a series of many small steps. Moreover, even after the replicatory structure of multiplicative relationships is discovered, these lower-attaining students may persist with counting-based arithmetical strategies for multiplication and division, and continue to visuospatially re-represent the units, such as with fingers or tally marks (Anghileri, 1989; Yeo, 2003), or a variety of other representational strategies (Finesilver, 2014).

There is considerable variation in the ease and manner in which individual learners develop the many aspects and levels comprising multiplicative thinking (Brown et al., 2010; Mabbott & Bisanz, 2008; Zhang, Xin, & Si, 2013), and thus in the level and kind of support required. While it is worth noting that certain currently-diagnosed neurocognitive divergences may affect mathematical development, the interplay of cognitive (e.g. perceptual or information processing impairment, etc.), affective and environmental factors contributing to an individual's mathematical difficulties are highly complex (Mulligan, 2011). Thus it is inappropriate to characterize those learners who struggle with school mathematics to a significantly greater degree than their peers (the definition of "low-attaining" used here) as anything other than a highly heterogeneous group with individual patterns of strength and weakness. Furthermore, this heterogeneity is the case for groups of individuals within any given neurodivergent condition (e.g. dyslexia, autism spectrum), who may or may not have low attainment in mathematics (Dowker, 2005; Yeo, 2003).

2D and 3D arrays as visuospatial representations of multiplicative structure

The 2-dimensional rectangular array representation is a standard one referred to in both current and previous versions of the National Curriculum for England and recommended for various activities requiring multiplicative thinking, such as solving one-stage multiplication and division tasks, or making explicit connections via visuospatially-represented number patterns between repeated addition and multiplication/division (DfE, 2014). It may be assumed that by the end of primary school (age 11), all students in English mainstream education will have encountered rectangular arrays (both dots and grids), and that these representations will have been employed in the context of multiplication. This pedagogical practice is supported by recent findings that the array form has particularly good potential for supporting reasoning and developing understanding in multiplication, and is one of the best for demonstrating the commutative and distributive principles (Harries & Barmby, 2007). Battista et al (1998) have identified various levels of sophistication in students' interaction with rectangular arrays of squares, and studies by Outhred and Mitchelmore (2000, 2004) have linked increased regularity of structure observed in young children's 2D array drawings to their development of multiplicative strategies.

While clearly a powerful tool for supporting the learning of multiplication (and also division), the 2D array provides limited enumeration options (e.g. for a 4×5 rectangle, working out either 4 rows of 5, 5 columns of 4, or simply unit-counting the 20 visible squares). With a 3D array, however, the enumeration options become more complex: for a cuboid with all dimensions >2 units, a one-dimensional strategy of unit-counting the visible cubes will no longer work, as there exist non-visible interior cubes; successful enumeration must then rely on conceptualizing the organizational structure of the array as a 3D space-filling object. While the expected final, formal strategy for students would be a three-way multiplication equivalent to the formula for the volume of a cuboid, on the way to this symbolic stage there are various potential strategies in which the cuboid structure is mentally deconstructed into manageable parts. For example, two possible deconstructions are to conceptualize the cuboid as (a) a one-dimensional arrangement of 2D horizontal *layers* (i.e. one thick vertical stack of rectangles), or (b) a two-dimensional arrangement of 1D vertical *columns* (i.e. many thin stacks laid out in a horizontal rectangular array).

A particularly influential series of writings on 3D array tasks (Battista, 1999, 2010, Battista & Clements, 1996, 1998) uses the concept of *spatial structuring*, which I adopt.

> We define spatial structuring as the mental act of constructing an organization or form for an object or set of objects. The process of spatial structuring includes establishing units, establishing relationships between units (such as how they are placed in relation to each other), and recognizing that a subset of the objects, if repeated properly, can generate the whole set (the repeating subset forming a composite unit). (Battista & Clements, 1996, p. 282)

According to Piagetian tradition, there are considered to be two types of experience required in the production of the mathematical understanding in this context: *physical* and *logico-mathematical* (e.g. Piaget, 1972). In the case of 3D arrays, the physical knowledge available to students would refer to the observable perceptual features of the blocks, in particular the way unit cubes are arranged in rows, columns and layers to make up the object. The logico-mathematical knowledge to be gleaned from this would be that successful spatial structuring of the block enables one to calculate how many unit cubes there are, and that neither orientation of the block, count order, nor count grouping affects the total. Note that while a child may pass the traditional Piagetian tests of order-irrelevance when applied to the enumeration of clearly separated discrete units (e.g. loose cubes, dot patterns), it cannot be assumed that that knowledge is automatically transferred to this alternative context, where the enumeration is of a continuous mass made up of smaller component units fixed into position, with no spaces, and including a subset of units which may be deduced but not directly viewed.

Regarding the presentation of 3D array tasks, prior to Battista and colleagues' research, Ben-Haim had studied children interpreting isometric drawings of blocks of cubes (e.g. Fig.2), in effect requiring participants to interpret tessellations of identical rhombuses as solid objects – a far from trivial requirement.

[INSERT Figure 2: Isometric drawing of cuboid]

Unsurprisingly, his original analytical framework reflected students' tendency to interact with the presented 2-dimensional image as a 2-dimensional shape ("1. counting the actual number of faces showing, 2. counting the actual number of faces showing and doubling that number" (Ben-Haim et al., 1985). Where more recent research by Battista and others has preferred perspective (Fig.1) over isometric projection, the interpretation difficulties present in line drawings of cuboids persist, as they have in the related tasks of Pitta-Pantazi and Christou (2010). Thus, there is impetus to observe whether, once the problematic requirement of interpreting 2D representations of 3D shapes is removed, and participants are simply presented with the solid shape itself, they display similar or different strategies and error patterns.

Counting in multiplicative tasks

The use of counting 'in ones' (verbally, with finger gestures, or both) as a main strategy in additive and multiplicative situations is generally associated with primary-age children, but persists through secondary school and indeed, adulthood as a supplementary or back-up strategy called upon for various reasons, for example at times when normal cognitive activity has been reduced - through tiredness, stress, or the presence of sensory (or other) distractions which disrupt fact retrieval and calculation processes, or when tasks involve non-decimal numerical structures such as time measures. However, there is a distinction to be made between retaining counting as a backup, and relying on it as a main enumeration strategy. Many studies (e.g. Geary, Bow-Thomas, & Yao, 1992; Gray & Tall, 1994; Ostad, 1997; Siegler, 1988) indicate that children with arithmetical difficulties are more likely than their typically-attaining peers to be reliant on counting-based strategies (as compared with, e.g. retrieval or derived fact strategies).

Anghileri (1995; 1997) describes a progression in children's counting in multiplicative scenarios, which while originally applied specifically to finger counting, applies equally well to other forms of counting with some kinesthetic and/or verbal component, beginning with 'unitary counting' (i.e. counting every item individually), through 'rhythmic counting' (discussed in detail below) to 'skip counting' (i.e. the disappearance of the interim numbers, leaving a sequence of multiples – also commonly called 'step counting').

This identification of a progression of increasingly-sophisticated stages for counting does not imply that a child passes categorically from one stage to the next, for, at any given time during development, an individual child may use a variety of arithmetical strategies for the same task (Baroody & Tiilikainen, 2003). Dowker goes further, suggesting that "Development consists not of the replacement of a single immature strategy by a single more mature strategy but of the discovery of increasingly more mature strategies, which co-exist for a long time with immature strategies" (Dowker, 2005, p. 22). While her research was mainly with younger children, results showing multiple strategy use have also been found for older children with low IQ scores (Baroody, 1988; Fletcher, Huffman, Bray, & Grupe, 1998).

When working with adolescents considered low-attaining in mathematics, one may reasonably expect to see a variety of developmental stages of counting-based strategies used by participants, and over the course of several tasks, likely a variety from any given participant.

Method

The data derive from tasks set during a series of individual or paired numeracy 'problem-solving interviews' run by the author as part of a larger research project, which employed microgenetic methods to observe and analyze emerging and developing multiplicative structure in low-attaining students' visuospatial representations (Finesilver, 2014). The problem-solving interviews also included a variety of other arithmetical activities, covered elsewhere, e.g. Cartesian product tasks in Finesilver (2009).

Microgenetic research design

Microgenetic methods were developed for the study of the transition processes of cognitive development (Siegler & Crowley, 1991). They have proved appropriate for case studies of individuals with difficulties in mathematics (Fletcher et al., 1998; Schoenfeld, Smith, & Arcavi, 1993) and have been used increasingly in studies of children's arithmetical strategies (e.g. Robinson & Dubé, 2008, 2009; Voutsina, 2012). The main characteristics, according to Siegler (2000, p.30) are:

- Observations span the period of rapidly changing competence.
- Within this period, the density of observations is high relative to the rate of change.
- Observations are analyzed intensively, with the goal of inferring the representations and processes that gave rise to them.

A frequent strategy in microgenetic research is to present a novel task and observe children's changing understanding of it across multiple sessions (Siegler & Crowley, 1991).

Schoenfeld et al. describe, metaphorically, a goal of taking "cognitive snapshots" (1993, p.61) each time a knowledge element or connection in their participant's knowledge structures changed; however, they also acknowledge that such moments were difficult to find. Thus more recent research in this tradition (e.g. Broza & Kolikant, 2015; Elia et al, 2014; Zhang et al, 2014) has incorporated elements from practitioner research and intervention studies, such as the inclusion of unique 'prompts' when no observable change is independently occurring, after which a new "cognitive snapshot" may be taken.

Participants

The thirteen participants were aged 11-15, and attending two mainstream secondary schools in inner London. Students had been proposed by their mathematics teachers for participation in the project on the basis of their history of, and ongoing, very low levels of numeracy compared to their peers. Those selected for participation were all registered with their schools as having Special Educational Needs (although this was not a stated selection criterion); with their individual files indicating a variety of diagnoses including dyslexia, autism spectrum, and global moderate learning difficulties. Most importantly, preliminary classroom observations, inspection of prior work, and their responses in initial sifting assessments indicated that, whatever the individual etiologies of their difficulties, the young people selected were all at a very particular stage of arithmetical development: that where one is familiar and comfortable with the principles of addition and subtraction but struggling with multiplication and division. I emphasize *principles* of addition and subtraction because, on the evidence sources stated above, these individuals made frequent errors in addition and particularly subtraction, and could not perform the operations outside of small-ish natural numbers; however, their understanding of the operations as metaphorically equivalent to combining and separating sets of objects was sound. In contrast, multiplications and divisions were mostly refused or only superficially attempted, with no evidence seen for sound metaphorical equivalences or intuitive models for these operations. As indicated in the literature above, this stage of arithmetical development is of particular relevance and interest.

Tasks

A 3D array task was set as the first activity in each of a series of four interviews, the first three of which were undertaken at 1-2 week intervals, the fourth later in the school year. Participants were presented with cuboid blocks made from $1cm^3$ multilink cubes, and asked how many cubes were present.

The blocks presented on the four occasions were:

1) One 3×4×5 cuboid (random mix of colors);
2) One 3×3×5 cuboid (as above);
3) Two 2×3×6 cuboids, one made from three differently-colored 2×6 layers, the other in six 2×3 layers; students were given the choice which to enumerate;
4) Two identical 2×2×3 cuboids, both colored in 2×3 layers; students were asked for the total number of cubes.

While this specific task type was new to these students, the representational media were familiar, as a previous task (set during initial sifting) had involved estimating and counting loose, unstructured quantities – a 'handful' – of the same cubes. Only the first block task was decided before the start of the study: specifics of each consecutive block were chosen based on the previous set of observed performances, as described below. No time constraint was imposed (with actual time taken varying from 1-15 minutes). Participants were allowed to handle the blocks but not take them apart, and were informed that the cuboid blocks were solid throughout.

Prompts

I began each tuition session with the question "How many of the small cubes make up this block?" If students were unable to work out a correct answer and requested help (or 'gave up' and ceased working), I provided input in the form of a series of minimal '*nudge*' prompts (Finesilver, 2014), intended to draw attention to specific structural characteristics.

Firstly, if a student was observed interpreting the task as counting the squares making up the surface area (rather than the cubes making up the volume) there were two dimension-related prompts: (a) producing a single loose cube, and reminding that these were the items to be counted; (b) indicating a vertex cube and demonstrating how they had counted it more than once.

The 'nudge' prompts of analytical interest in this paper (relating to multiplicative thinking) were designed to draw students' attention to the internal repeating layers of the cuboid; that is, that however many cubes were in the top layer, the same number would be found underneath, and again underneath that. The intention was that the students should notice the replication inherent in the cuboid structure, and use reasoning to develop an appropriate enumeration strategy using available information.

The 'layers' prompts, always deployed in the same order, were:

a) Enquiring how many cubes made up the top layer;
b) Enquiring how many were in the layer underneath (and, if necessary, the one beneath that);
c) Stating explicitly that all layers contained the same numbers of cubes;
d) Stating the numbers in each layer in the form of an addition (and, if necessary, supporting or performing that calculation).

These micro-interventions were designed to prompt a layers-based spatial structuring. This is not because it is thought to be intrinsically better or more advanced than a columns-based structuring; it was because (a) professional experience of using this exact task in mainstream and special education classrooms had indicated it as the most common by far, and (b) it leads to the calculation 20+20+20 (or 3×20, which has many fewer steps than the alternative, with a 'round' number that would be comparatively unchallenging for these students.

Data collection

Audio and visual data were collected and triangulated via recordings from a microphone left continuously running, all paper marked by students or researcher during the sessions, and field notes describing students' observed actions.

Analytical framework

One of the main outcomes of the larger research project was the development of analytical frameworks suitable for detailed qualitative analysis of students' visuospatial representational activity. While the primary aim of this sub-project was the documentation of low-attaining students' multiplicative thinking in concretely-presented array tasks, a secondary aim became to build on previous research to produce and test a framework that would allow for more detailed analysis of student activity in 3D array tasks, particularly when these are presented concretely rather than in diagram form. Thus the data from this small but diverse group were treated under three distinct but connected analytical aspects: *spatial structuring* (how the physical structure of the presented object is conceptualized by participants, and the corresponding numerical relationships drawn from it for use in the enumeration process); *enumeration* (exactly how participants make use of the numerosities they derive from the presented object), and *errors* (the ways participants' interactions with arrays and subsequent enumeration strategies may go wrong).

The *spatial structuring* aspect of the framework (see Table 1) is based on that of Battista and Clements (e.g. 1996), who classified their students' interactions with 3D arrays into a set of categories and subcategories. My adaptations were to alter the descriptors to apply to actual physical cuboids (as opposed to 2D images), to restructure them within a loose hierarchy, and to expand the category structure to include certain theoreticallygenerated strategies that, while not actually occurring in the current dataset, are plausible (e.g. C1).

[INSERT Table 1: Spatial structurings of a 3D array]

The *enumeration* aspect (see Table 2) is based on the counting classification used by Anghileri (1997). By assigning the numbers 1 for multiplication, 2 for addition and 3 for counting, these may be combined with the spatial structuring categories in the previous section (producing e.g. *C3g* to describe grouped counting that is structured by columns). My main alteration concerns the category Anghileri (and others) have termed 'rhythmic counting', which I consider to be the conflation of two distinct count types that ought properly to be separated for incremental developmental analysis.

To explain this distinction, consider a child conducting a verbal count of some units that have been arranged in groups of three, e.g.:

1.2.3…...

4.5...6….....

7.…8...9……

where each group of number words is delineated from the next by a perceptible temporal separation (analagous to the spatial separation between visible groups of units). However, inter- and/or intra-group gaps between number words items are not temporally evenly spaced, and do not have any particular emphasis (e.g. louder speech, larger gesture) on the cardinal number of each group. This may be termed 'grouped counting'. On the other hand, a verbal grouped count might also have a regular rhythmic structure to it, e.g.:

1.2.**3**…

4.5.**6**…

7.8.**9**…

where both the gaps between individual numbers verbalized and between groups are produced in a regular temporal pattern at constant tempo (analogous to the way that units in array representations are positioned in a regular spatial pattern). The final number of each group has perceptibly greater verbal or gestural emphasis. The term 'rhythmic counting' should be reserved for the latter type alone, and I suggest that this musical 'drive' particularly aids the subsequent move to step counting. (Note that these categories do not just apply to verbal counting: arrhythmic and rhythmic grouping may also be indicated and distinguished by, say, tapping cubes with a finger).

[INSERT Table 2: Enumeration strategies for a 3D array]

The *errors* aspect (see Table 3) includes four potential error types, based on the aspects already discussed, and also data collected in this project. The possibility of students attempting to enumerate based on an incorrect spatial structuring (e.g. Faces) has already been discussed. Given the importance of the move from counting to calculation, it seems appropriate diagnostically to separate errors made by students who did and did not attempt to carry out a numerical calculation for their enumerative strategy. Finally, the error type I describe as *visuospatial/kinesthetic* was observed amongst participants in this study, but has not to my knowledge been noted in prior research on this topic.

[INSERT Table 3: Types of error when enumerating 3D arrays]

Results

A coding of all individual participant responses to all tasks may be found in the Appendix. This section, in microgenetic tradition, first provides a narrative description (summary with highlighted examples) of the various participant responses over time, as a window onto the various perceptual and cognitive processes observed.

Task 1: Initial responses to a 3D array (3×4×5)

When presented with their very first block, all students used some form of counting-based strategy, and every one gave an incorrect answer.

Two students (Ellis and Wendy) independently made perceptive, effective use of one deconstruction of the array structure, in a strategy which would have been successful had they not then made minor counting errors which delivered answers of 59 and 61 rather than 60. With the block on the table, they placed a finger on one of the cubes in the top (4×5) layer and said "1, 2, 3", referring to the touched cube and the two that were vertically beneath it, then moved the finger along one cube, saying "4, 5, 6", continuing to (arhythmically) group-count threes for every cube in the top layer of 20, with a verbal count error for Ellis (repeating a number name) and a visuospatial/kinaesthetic error for Wendy (repeating a finger tap). It is notable that neither student gave any indication of recognizing the cardinal numbers of each count-group as the set of multiples of three, which is consistent with lack of rhythmicity, and with not noticing their respective miscounts.

Ten of the remaining students began by either unitary-, group- or step-counting the top layer, then moved onto the other faces of the block, turning it around and attempting to count all the external cubes. Although some students specifically asked for confirmation that the shape was solid as opposed to hollow, their face-based counting strategies nevertheless ignored non-visible interior cubes. Meanwhile, the lack of clear points at which to start and stop counting, or of an obvious 'route' around the six faces, also led to some cubes and/or whole faces being counted more than once, while others were missed out. Close observation of gestures and comments indicated that four of these students were attempting to avoid double-counting, but the other six gave no sign of noticing that they had double-counted edge cubes or triple-counted vertex cubes. This latter indicates a more serious spatial structuring error: they were interacting with a 2-dimensional space instead of a 3-dimensional one, counting the component squares of the cuboid's surface area instead of the component cubes of its volume.

One student gave an unexpectedly idiosyncratic initial response to the task. Like most others, Leo did not make use of the cuboid structure of the block on his initial attempt, and also made enumeration errors. The atypicality is that he organized his counts by color, first counting all the visible red cubes and writing the subtotal in red, then doing the same for blue, green and black. He ignored all other colors.

[INSERT Figure 3: Colours and calculation (Leo)]

When I enquired about the yellow, pink and brown cubes he stated that he "didn't put them in". Leo (who has an autism spectrum diagnosis) was particularly attached to his favorite pen – a four-color ballpoint; it appears that using only his preferred pen was of higher importance to him than totaling the remaining cubes. (Writing a number down in a non-matching ballpoint color, or using my proffered felt-tips were unacceptable options to him.) This example of a neurodivergent student's unusual priorities in carrying out a task – picked up only by this type of close 1:1 observation – was reinforced by his decision to invent his own (incorrect) form of multi-digit addition (Fig.3).

Task 1: First responses to prompts

Of the 13 students, six responded to one of the first three 'nudge' prompts by stating the number of cubes in each layer and then calculating a total of 60. Three more required the full demonstration/explanation, but then indicated verbally that they understood. One (Paula) gave no indication that she understood either my addition of three twenties (written in standard vertical notation) or its relevance to finding the total number of cubes.

The two students who had initially used a columns-based grouped-counting strategy were confident enough in their conceptualization and strategy that it could reasonably be expected that they would not be confused by discussion of an alternative deconstruction, and indeed they responded positively to my introduction of a layers-based alternative. One, Wendy, became immediately engaged with the idea of different ways to get to an answer.

CF: *How many are just in the top layer?*

W: *20.*

CF: *Yeah. So if there's 20 just in that top layer, and then there's exactly the same underneath it –*

W: *Oh! So that would be 40, then 60.*

CF: *So we could say there's . . . three layers of 20, which is 60. Or, if you happened to have it up a different way [rotates block], how many are in the top layer now?*

[We work through the process for five layers of twelve.]

CF: *Or if we happened to have it up that way [rotates] to start with –*

W: *15.*

CF: *And how many layers of 15?*

W: *4.*

CF: *So whatever way up it happened to be –*

W: *Still be 60!*

Meanwhile, Leo's response to the single prompt (a) indicates how potentially effective a minimal 'nudge' can be in altering a student's thinking about a task, even a student who, as seen, has some atypical motivations and mathematical behaviors.

CF: *How many are in just the top layer?*

[Leo step-counts 5, 10, 15, 20, and affirms that there are 20 in the top layer.]

L: *Ah!*

CF: *Does that help at all with getting the total number?*

L: *Well now I think I have a solution to this!*

CF: Ok.

L: If you were able to split this, if you chop the layers off, it'll be 20 there... underneath that is another 20, and underneath that is another 20.

[Leo starts drawing – see Figure 4.]

Leo: That's 20 there and 20 there. You could just pull it out like a drawer, then pull that out like a drawer. It would be 20, 20, 20!

[INSERT Figure 4: 'Drawers' (Leo)]

Task 2 (3×3×5 array)

Having decided against increasing the numbers involved, all students were now set a similar but slightly smaller cuboid, with the intention of observing whether they would replicate the successful strategies they had used or observed in the previous session, and if any alternative strategies would appear. Prompts were deployed as before, only where needed.

Two students replicated a full layers strategy correctly and independently. Three others (including Leo) began appropriately by counting and stating the number in the top layer, but still needed a prompt to complete the enumeration process. Of the two who had used a columns strategy the first time, Ellis counted the same way but with noticeably increased rhythmicity (i.e. a change from strategy *C3g* to *C3r*), while Wendy chose horizontal instead of vertical columns to organize her counting. Four students initially reverted to the incorrect strategy of counting around the faces, but switched quickly to layers on one or more prompts. Paula made no attempt to enumerate, via faces or otherwise, and still gave no discernible sign of understanding the demonstrated process.

Task 3 (2×3×6 array)

After two tasks, only four students could be said to be confidently carrying out appropriate strategies; one student gave no indication of understanding even complete demonstrations, and the other eight were at some stage of partial understanding and operationalisation. (Note that 'appropriate' here does not imply efficient, or even that the student gave a correct answer – only that the steps taken, if implemented without errors, would produce the correct total.) Hence, for Task 3 I employed the cubes' color to highlight the physical structure, constructing blocks with each layer a different color. Rather than force students into one particular color structure (and thus a particular numerical decomposition), I let them choose between two equal-sized blocks: a 3-colour block in horizontal 2×6 layers or a 6-colour block in vertical 2×3 layers (or pairs of columns). This was intended to increase the chances of struggling students perceiving and using the equal layers structure, while also providing opportunity for the more confident students to perhaps discover and comment (as Wendy had) on alternative potential calculations of the total. For this and Task 4 I also made Paula her own special blocks with smaller numbers of larger-sized cubes, for ease of both arithmetical calculation and handling.

This time 8 students used the layers structure in their initial attempts at calculation, with only two errors observed (numerical calculation, verbal count). Only three students' initial response was still face-based, with Paula for the first time able to comprehend and work through the task (with full set of prompts).

The colored layers were indicated to be helpful by student comments, such as "you don't get confused", and "if there are 6 cubes there [i.e. in an end layer] then you know there's six in the rest [of the layers]", although it should be noted that Ellis ignored the colored layers and retained his trusted columns-based rhythmic count. The 3-colour block was chosen by nine students, and the 6-colour by four; implications of these choices are discussed later.

Task 4 (two 2×3×3 arrays)

As most students were now enumerating the cubes in an appropriate, systematic fashion with little or no support, for their final array task I introduced an additional structural aspect: students were presented with two identical color-layered blocks, and asked for the total number (i.e. from both). This again served a dual purpose: for the students still struggling, it was an opportunity for a second attempt at enumerating the more manageable color-structured blocks, while, with a numerical structure of 2(2×3×3) – i.e. a 4-dimensional multiplication – there were further increased calculation possibilities to stretch the more confident students.

Ten of the students enacted an appropriate layers-based strategy without any strategic errors, and with increased attempts at step-counting, addition or multiplication rather than counting every cube – resulting in an increase in calculation and retrieval errors. The two weakest students still first attempted a faces-based unitary count then used layers successfully with prompts. All students demonstrated interaction with the various replicatory structures of the cuboid – mostly the colored layers, but with some spatial words and/or gestures referring to horizontal rows and vertical columns.

Regarding the duplicate blocks: Five students used some form of counting (unitary-, grouped- or step-) for the first block, then repeated their actions (continuing the count) for the second. Two students pushed the two blocks together and treated them as single mass. Five students worked out that there were 18 cubes in the first block and doubled (or added another 18) for the total. (One more stated that he thought of doing this, but was unconvinced that the two blocks were really the same, and insisted on counting the second before adding.)

Discussion

Spatial structuring

Apart from the two students who immediately perceived the columns, initial responses to the task showed little or no awareness of the block's three-dimensional array structure. Students interacted with faces only, one face at a time, failed to coordinate orthogonal views from different perspectives, and in many cases did not even have a complete faces-based conceptualization of surface area. In fact, their strategies did not greatly differ from those used when enumerating a loose pile of cubes – with the inconvenience that counted cubes could not be physically separated from uncounted ones.

All students showed increased awareness and use of physical structure following interviewer prompts. For some students, a minimal nudge prompt enabled immediate perception of the 3D spatial structure, which was retained throughout subsequent tasks. However, others required substantial explanation and/or demonstration, and although increased understanding was indicated by their layers-based enumerative activity, gestures and verbalizations, they later reverted to faces-based strategies one or more times.

Over the four tasks, there appeared to be a general move from faces towards layers-based structurings, as expected given the choice of prompts, and use of color in Tasks 3-4. For those with greater difficulties, the use of colored layers in Tasks 3 and 4 made a significant difference, conforming with Battista's suggestion that "using color . . . might promote the perceptual integration that supports conceptual integration" (2010, p. 196). For Paula (by far the weakest participant), it proved particularly effective to link concrete and symbolic representations through the use of matched colored pens for recording the number in each layer (see Figure 5).

[INSERT Figure 5: 3-colour block, 3-colour sum (author and Paula)]

On finding a successful strategy, some students repeated it in precisely the same way for each task, whereas others experimented with strategies based on different structural aspects (layers, columns, rows, and combinations of these). This variability aligns with Dowker's and others' evidence for strategic variability in arithmetic, but not with certain of Battista and Clements' comments about their model, which assume a more straightforward strategic progression. They have described layers-based strategies as indicating that students "see the array as space-filling" and have "completed a global restructuring of the array" (1998, p. 234). Meanwhile, columns-based strategies were characterized thus: "Those in transition, whose restructuring was local rather than global . . . They had not yet formed an integrated conception of the whole array" (ibid). It is unclear why a columns-based spatial structuring should be considered any less 'global' or 'integrated' than a layers-based one. The former deconstructs a 3D array into a 2D array of 1-dimensional stacks, the latter a 1-dimensional stack of 2D arrays; both are equally valid as 'space-filling conceptions'. In most (but not all) cases, using layers results in a more efficient calculation than using columns, but this must not be confused with it being a more advanced conceptualization of the array structure. Here, and in other non-time-pressured settings, students electing to test out alternative strategies – even when less efficient – should be considered evidence of more, not less, integrated conceptualization.

Enumeration

On encountering Task 1, all students used some form of counting, and overall, counting-based strategies were by far the most popular. Four were observed definitively using a multiplication calculation (number in a layer × number of layers) in either Task 3 or 4 (i.e. when the layers were defined by color), and there were occasions where language used implied recognition of multiplicative structure (e.g. a student verbalized "two twelves", although did not produce the answer 24). However, in the strategic space between unitary counting and multiplication was observed a varied spectrum of ad-hoc grouped, rhythmic, and step-counting, and also repeated addition. Students used different enumerative strategies from task to task, and sometimes mixed them within the same task.

The previously-mentioned analytical issue of distinguishing rhythmic counting from (arrhythmic) grouped counting may be illustrated further by two examples:

(1) Ellis had begun by counting aloud all individual cubes, organized by column. Then, as his counting increased in rhythmicity, he ceased verbalizing the non-cardinal numbers of each subgroup, and represented them kinesthetically – tap, tap, 3, tap, tap, 6, etc. – a clear interim stage in the progression to full step-counting.

(2) Tasha had group-counted a block in horizontal rows of three, only hovering a finger vaguely above the block rather than tapping or pointing to each individual cube, and had not noticed that one of her groups had only 2 in it, giving a subtotal of 17 rather than 18. When she re-counted, pointing at each top cube with exaggerated rhythmic drive, all groups now contained three numbers, and the multiples of 3 also received greater vocal emphasis, highlighting their importance.

Errors

Spatial Structuring: Issues of spatial structuring have been covered in detail above. To summarize, while all but two students' initial responses to Task 1 involved mis-structuring, instances of spatial structuring error by all 13 participants on the following three tasks numbered only 9 in total; this is a drastic reduction.

Numeric Calculation: The predominant preference of low-attaining students for counting-based strategies meant that recall of arithmetical facts or procedures was not often attempted (although there were indications it might increase with task familiarity). In total, on nine occasions, students mis-recalled addition facts and number patterns, or unsuccessfully attempted formal 'vertical' addition notation for the layers.

Verbal Count: All participants were confident in their ability to count individual cubes, but this confidence was sometimes misplaced. For example, while counting aloud, Ellis was heard repeating a number and Paula skipping a decade without noticing.

Visuospatial/Kinesthetic: The most common error type, 22 instances of this were observed, with two students making them on all four occasions. Some were gesture-related, such as the students with weaker fine motor skills moving their finger at a different speed to their verbal count, or taking too large a 'jump' and skipping one or more cubes, rows, columns or layers. Other errors concerned the handling of the block, such as when students using faces-based strategies over-rotated the cuboid. (This last error was corrected by improved spatial structuring: once a columns and/or layers conceptualization is achieved, the block can remain immobile.)

Different strategic improvements were effective in counteracting different errors. For example, Tasha's move to rhythmic counting helped precisely because her error had been of the visuospatial/kinesthetic type; it would not have corrected a verbal count error. A tendency to different error types may be another factor in explaining why some students preferred columns over layers, and the 6-colour block over the 3-colour block (or vice versa).

Potential complications with classification

As with the previous frameworks on which it builds, this system is not without classification issues. First, a combination of strategies may be deployed within the same task (e.g. adding the first two layers then unit-counting the next). In some cases the observer may have to look very closely to identify the strategy a student is using (e.g. a silent student giving mostly correct answers, but without the verbal capacity to explain strategies coherently, meaning they must be inferred from gestural data alone). Alternatively, with students who do verbalize their working, there may be inconsistencies between what they report doing and what they are observed doing (e.g. identifying a multiplication such as "it's three twelves", but, failing to retrieve the answer to 3×12, actually working it out by counting). From this evidence, it is likely that some other students, who did not tend to verbalize their thought processes, could perceive the multiplicative structure yet were unable or unwilling to carry out the multiplication operation otherwise than by counting.

Conclusions

Low-attaining students engaging with concretely-presented 3D array tasks

Past research on 3D arrays (e.g. Battista & Clements, 1998; Ben-Haim et al., 1985) has demonstrated various types of incorrect and partial spatial structurings frequently used by students in their enumerative efforts, particularly where 2D images were used to represent 3D arrays. Similar findings were observed in this study; despite having an array that could be picked up and rotated, full 3D spatial structuring was initially absent in most cases.

Past research on enumeration (e.g. Anghileri, 1989, Dowker, 2005) suggests that one can expect students with difficulties in mathematics to tend strongly towards counting-based strategies. This was also confirmed; however, there was notable variety in the counting styles, sub-stages, and error patterns that is not often acknowledged within the 11+ age group.

When informing students that their task solutions are incorrect, one hopes for the kind of cognitive conflict which results in critical reflection and adaptation of strategies. This did not happen here, so strategic progression required external input. The amount of prompting required and strategic change observed varied widely, as expected from Brown et al.'s (2010) and Zhang et al.'s (2013) work on multiplicative reasoning development; however, encouragingly, in some cases that input was minimal when considering the qualitative change produced. Even for those students struggling the most, indications of potential progression were observed. (This conclusion is necessarily tentative, due to methodological limitations).

Relationship between spatial structuring and enumeration in developing multiplicative thinking

Interacting with visually-perceptible replicatory patterns within concrete structures can increase awareness of the multiplicative numeric patterns they embody. While Outhred and Mitchelmore (2000, 2004) have observed this in 2D array drawings, it can also be seen when a student is pointing with a finger to unit-count cube arrays, and the physical motion required to move from one set to the next causes a pause in count sequence, naturally grouping the counting numbers and emphasizing the last (cardinal) of each group, thus increasing rhythmicity – an important step towards step-counting. However, this is not clear-cut diagnostically; it may be that use of columns of three (for example) in a given student's counting does not necessarily entail a realization that three has an integral role in the array structure, if it is simply seen as a short-cut for counting. Similarly, the converse is observed, where students show awareness of the physical replicatory structure but this is not reflected in their enumeration.

Regarding causality, while it is tempting to assume that students' enumeration of 3D arrays is a direct result of their spatial structuring – and initially this might be so – the ongoing relationship is actually bidirectional. Once a student is familiar with 3D array tasks, enumeration can also guide structuring (i.e. the particular way a given 3D object is deconstructed into manageable components). For example, an individual who prefers to step-count long sequences of small quantities than to add a short sequence of large quantities may opt for columns over layers. Thus, awareness of alternative structurings is beneficial, as individuals can can choose that which best suits their enumerative capabilities and preferences. Furthermore, the flexibility to switch pragmatically between different array structurings can support understanding of further properties of multiplication (e.g. associativity or commutativity, as recommended by Harries & Barmby (2007)).

Utilizing 3D array tasks in research and teaching

Learners' engagement with the multiplicative structures instantiated in 3D arrays is complex and nuanced, particularly for those whose arithmetical attainment is delayed or impaired. For better understanding of individuals' performance and progression on this kind of mathematical task, I have taken two frameworks for analyzing spatial structuring in arrays and enumeration in multiplicative situations (respectively) and demonstrated the potential benefits of using both in combination, and increasing the set of categories for each.

3D array tasks used in this way prove rich in diagnostic information on various factors involved in learners' developing multiplicative thinking. A microgenetic approach to observation is effective for sensitively differentiating kinds of spatial structuring and enumeration, and particularly between the error types causing incorrect responses. This latter has direct pedagogical implications, as the nature of support one might provide a student would be different depending whether their difficulty was, for example, in perceiving the array's replicatory structure, in carrying out counts or calculations, or caused by inadequate motor skills for keeping their finger in place.

To conclude, although the layered spatial structure of a cuboid seems obvious to the majority of adults and many typically-attaining students of age 11 up, there are clearly some within mainstream education who struggle significantly. Yet, with appropriate guided activity they can conceptualize the 3D array as a coordinated, space-filling structure. Further, some can experience the 'aspect shift' of perceiving multiple alternative structurings, and while this may be unnecessary in the short term (i.e. for solution of this particular task), in the wider aim of developing flexible multiplicative thinking, it is advantageous and to be encouraged.

References

Anghileri, J. (1989). An investigation of young children's understanding of multiplication. *Educational Studies in Mathematics, 20*(4), 367–385.

Anghileri, J. (1995). Children's finger methods for multiplication. *Mathematics in School*, *24*(1), 40–42.

Anghileri, J. (1997). Uses of counting in multiplication and division. In I. Thompson, (Ed.), *Teaching and learning early number* (pp. 41–51). Maidenhead: Open University Press.

Baroody, A. J. (1988). Mental-addition development of children classified as mentally handicapped. *Educational Studies in Mathematics*, *19*(3), 369–388.

Baroody, A. J., & Tiilikainen, S. H. (2003). Two perspectives on addition development. In *The development of arithmetic concepts and skills: Constructing adaptive expertise* (pp. 75–125). Mahwah, NJ: Lawrence Erlbaum Associates.

Battista, M. T. (1999). Fifth graders' enumeration of cubes in 3D arrays: Conceptual progress in an inquiry-based classroom. *Journal for Research in Mathematics Education*, *30*(4), 417–448.

Battista, M. T. (2010). Thoughts on elementary students' reasoning about 3d arrays of cubes and polyhedra. In Z. Usiskin, K. Andersen, & N. Zotto (Eds.), *Future curricular trends in school algebra and geometry: proceedings of a conference*, 183–202.

Battista, M. T., & Clements, D. H. (1995). Enumerating Cubes in 3-D Arrays: Students' Strategies and Instructional Progress. *Proceedings of the 17th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* Vol. I, 192-198.

Battista, M. T., & Clements, D. H. (1996). Students' understanding of three-dimensional rectangular arrays of cubes. *Journal for Research in Mathematics Education*, *27*(3), 258–292.

Battista, M. T., & Clements, D. H. (1998). Students' understanding of three-dimensional cube barrays: Findings from a research and curriculum development project. In R. Lehrer

& D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 227-248). London: Routledge.

Battista, M. T., Clements, D. H., Arnoff, J., Battista, K., & Borrow, C. V. A. (1998). Students' spatial structuring of 2D arrays of squares. *Journal for Research in Mathematics Education*, *29*(5), 503–532.

Ben-Haim, D., Lappan, G., & Houang, R. T. (1985). Visualizing rectangular solids made of small cubes: Analyzing and effecting students' performance. *Educational Studies in Mathematics*, *16*(4), 389–409.

Brown, M., Küchemann, D., & Hodgen, J. (2010). The struggle to achieve multiplicative reasoning 11-14. In *Proceedings of the Seventh British Congress of Mathematics Education,* Vol. 30, 49–56.

Broza, O., & Kolikant, Y. B.-D. (2015). Contingent teaching to low-achieving students in mathematics: challenges and potential for scaffolding meaningful learning. *ZDM*, *47*(7), 1093–1105.

Clark, F. B., & Kamii, C. (1996). Identification of multiplicative thinking in children in Grades 1-5. *Journal for Research in Mathematics Education*, *27*(1), 41–51.

Department for Education (DfE). (2014). *National curriculum in England: mathematics programmes of study*. Retrieved 3 June 2015, from https://www.gov.uk/government/publications/national-curriculum-in-england-mathematics-programmes-of-study

Dowker, A. (2005). *Individual differences in arithmetic: Implications for psychology, neuroscience and education*. Hove: Psychology Press.

Elia, I., Gagatsis, A., & Heuvel-Panhuizen, M. van den. (2014). The role of gestures in making connections between space and shape aspects and their verbal representations in the early years: findings from a case study. *Mathematics Education Research Journal*, *26*(4), 735–761.

Finesilver, C. (2009). Drawing, modelling and gesture in students' solutions to a Cartesian product problem. *Educate~*, *9*(3), 21–36.

Finesilver, C. (2014). *Drawing division: emerging and developing multiplicative structure in low-attaining students' representational strategies*. PhD thesis, Institute of Education, London. Retrieved from http://eprints.ioe.ac.uk/20816/

Fletcher, K. L., Huffman, L. F., Bray, N. W., & Grupe, L. A. (1998). The use of the microgenetic method with children with disabilities: Discovering competence. *Early Education & Development*, *9*(4), 357.

Geary, D. C., Bow-Thomas, C. C., & Yao, Y. (1992). Counting knowledge and skill in cognitive addition: A comparison of normal and mathematically disabled children. *Journal of Experimental Child Psychology*, *54*(3), 372–391.

Gray, E. M., & Tall, D. O. (1994). Duality, Ambiguity, and Flexibility: A 'Proceptual' View of Simple Arithmetic. *Journal for Research in Mathematics Education*, *25*(2), 116–140.

Harries, T., & Barmby, P. (2007). Representing and understanding multiplication. *Research in Mathematics Education, 9*(1), 33–45.

Izsák, A. (2005). 'You have to count the squares': Applying knowledge in pieces to learning rectangular area. *The Journal of the Learning Sciences*, *14*(3), 361–403.

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.

Mabbott, D. J., & Bisanz, J. (2008). Computational Skills, Working Memory, and Conceptual Knowledge in Older Children With Mathematics Learning Disabilities. *Journal of Learning Disabilities*, *41*(1), 15–28.

Mulligan, J. (2011). Towards understanding the origins of children's difficulties in mathematics learning. *Australian Journal of Learning Difficulties, 16*(1), 19–39.

Ostad, S. A. (1997). Developmental differences in addition strategies: a comparison of mathematically disabled and mathematically normal children. *British Journal of Educational Psychology*, *67*(3), 345–357.

Outhred, L. N., & Mitchelmore, M. C. (2000). Young children's intuitive understanding of rectangular area measurement. *Journal for Research in Mathematics Education*, *31*(2), 144–167.

Outhred, L.N, & Mitchelmore, M. (2004). Students' structuring of rectangular arrays. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, 465–472.

Piaget, J. (1972). Intellectual Evolution from Adolescence to Adulthood. *Human Development*, *15*(1), 1–12.

Pitta-Pantazi, D., & Christou, C. (2010). Spatial versus object visualisation: The case of mathematical understanding in three-dimensional arrays of cubes and nets. *International Journal of Educational Research*, *49*(2–3), 102–114.

Robinson, K. M., & Dubé, A. K. (2008). A microgenetic study of simple division. *Canadian Journal of Experimental Psychology*, *62*(3), 156–162.

Robinson, K. M., & Dubé, A. K. (2009). A microgenetic study of the multiplication and division inversion concept. *Canadian Journal of Experimental Psychology*, *63*(3), 193–200.

Schoenfeld, A. H., Smith, J. P., & Arcavi, A. (1993). Learning: The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In *Advances in instructional psychology* (Vol. 4, pp. 55–175). Hillsdale, NJ: Lawrence Erlbaum Associates.

Sherin, B., & Fuson, K. (2005). Multiplication strategies and the appropriation of computational resources. *Journal for Research in Mathematics Education*, *36*(4), 347–395.

Siegler, R. S. (1988). Individual differences in strategy choices: good students, not-so-good students, and perfectionists. *Child Development*, *59*(4), 833–851.

Siegler, R. S. (2000). The rebirth of children's learning. *Child Development*, *71*(1), 26–35.

Siegler, R. S., & Crowley, K. (1991). The microgenetic method : A direct means for studying cognitive development. *American Psychologist*, *46*(6), 606–620.

Voutsina, C. (2012). A micro-developmental approach to studying young children's problem solving behavior in addition. *The Journal of Mathematical Behavior*, *31*(3), 366–381.

Watson, A. (2006). *Raising Achievement in Secondary Mathematics*. Maidenhead: Open UP.

Yeo, D. (2003). *Dyslexia, dyspraxia and mathematics*. London: Whurr.

Zhang, D., Xin, Y. P., Harris, K., & Ding, Y. (2014). Improving Multiplication Strategic Development in Children With Math Difficulties. *Learning Disability Quarterly*, *37*(1), 15–30.

Zhang, D., Xin, Y. P., & Si, L. (2013). Transition From Intuitive to Advanced Strategies in Multiplicative Reasoning for Students With Math Difficulties. *The Journal of Special Education*, *47*(1), 50–64.

Appendix

	Task 1: mixed-colour cuboid, 3x4x5	**Task 2:** mixed-colour cuboid, 3x3x5	**Task 3:** choice of striped cuboids, 2x3x6	**Task 4:** twin striped cuboids, each 2x2x3
Ellis	C3g VC error	C3r -	C3r * -	L3g * -
Wendy	C3g VK error	L3g VK error	L3g ** -	L3g -
Jenny	F3u (cubes) SS, VK errors	L2 -	L3g -	L2 (2nd block L3g) -
Tasha	F3g (cubes) SS, VK errors	L2 NC error	L3r -	L3r VK error
Sidney	F3s (cubes) SS, VK errors	L3u VK error	L1 VK error	L2 (2nd block L3s) NC error
Leo	O (colours) SS, NC errors	L2 NC error	L2 -	L1 * -
Kieran	F3u (squares) SS, VK errors	L3 -	L3 -	L2 -
Danny	F3s (squares) SS error	F3s (cubes) SS error	L1 ** -	L2 (NC error on 2nd)
George	F3g (squares) SS, VK errors	*[Recording malfunction]*	L3g -	L3g/s (NC error on 2nd)
Oscar	F3g (squares) SS, VK errors	F3u (cubes) SS, VK errors	C2 *** NC error	C2 *** -

Harvey	F3g (squares) SS, VK error	F3s (cubes) SS, VK, NC errors	F3g (cubes) SS, VK errors	L1 (2nd block L3) (VK error on 2nd)
Vince	F3g (squares) SS, VK errors	F3g (cubes) SS, VK errors	F3g (cubes) SS, VK errors	F3u (cubes) SS, VK errors
Paula	F3u (squares) SS, VC, VK errors	No attempt	F3u (squares) SS, NC errors	F3u (squares) SS, VK, NC errors
Layers	0	6	8	10
Columns	2	1	2	1
Faces	10	4	3	2

Appropriate strategy carried out correctly

Correct spatial structuring with error(s) in carrying out enumeration (NC, VC VK)

Faces-based structuring, attempting to count cubes making up volume (SS)

Faces-based structuring, attempting to count squares making up surface area (SS)

* On Tasks 3, Ellis refers verbally to layers but counts in columns. On Task 4, he used vertical layers, subdivided into columns.

** On Task 3, Wendy and Danny used layers, but not those delineated by colour

*** On Tasks 3-4, Oscar appeared to be initially spatially structuring in columns, then mentally grouping these into vertical layers for addition.

Tables 1-3 [For insertion into main text]

Table 1: Spatial structurings of a 3D array

M	**The student conceptualises the set of cubes as a 3D multiplicative structure**	
		Student determines the length, width and height of the block, and multiplies the three numbers.
L	**The student conceptualises the set of cubes as forming a stack of 2D layers**	
	1	*Layer multiplication*: Student computes or counts the number of cubes in one (usually horizontal) layer, counts the number of layers, and multiplies the two numbers.
	2	*Layer addition*: Student computes or counts the number of cubes in one layer and uses addition or step-counting (indicating successive layers) to get total.
	3	*Counting subunits of layers*: Student's counting of cubes is organized by layers, but the student unit-counts or step-counts by a number smaller than the number of cubes in a layer
C	**The student conceptualises the set of cubes as forming a 2D array of columns**	
	1	*Column multiplication*: Student counts the number of cubes in one (usually vertical) column, counts the number of columns, and multiplies the two numbers.
	2	*Column addition*: Student counts the number of cubes in one column and uses addition or step-counting (indicating successive columns) to get total.

	3	*Counting subunits of columns*: Student's counting of cubes is organized by columns, but the student unit-counts or step-counts by a number smaller than the number of cubes in a column.
F	**The student conceptualises the set of cubes in terms of its faces**	
		Student counts around the faces of the cuboid. They may be counting cubes or counting squares. (Either way, this will almost always be incorrect, as it would only be possible to obtain a correct answer by taking account of all cubes appearing on more than one face, and adding on the unseen interior cubes.)
O	**Other**	
		Student uses a conceptualisation other than those described above.

Table 2: Enumeration strategies for a 3D array

1	**Multiplication**
	Student calculates a total without any indication of addition or step-counting (i.e. uses direct retrieval of the multiplication facts involved, or a calculation strategy based on associated retrieved multiplication facts).
2	**Addition/Step-counting**
	Student counts in steps formed of the cardinal number of each layer or column, without indicating any interim numbers (i.e. uses a number pattern based on addition facts).
3	**Counting**
s	Count includes step-counting of some cubes (when used *within* a layer)
r	Rhythmic counting: Student counts each cube individually, but the count sequence is rhythm-driven, with clear emphases on the cardinal number of each (equal) subgroup.
g	Grouped counting: Student counts each cube individually, but with the count sequence organised into subgroups.
u	Unitary counting: Student counts each cube individually, without any grouping.

Table 3: Types of error when enumerating 3D arrays

Spatial structuring (SS) Student uses an incomplete or incorrect conceptualisation of the array structure, e.g. double-counting edge cubes, not accounting for interior cubes.
Numeric calculation or retrieval (NC) Student makes an error in calculating or retrieving a number fact while multiplying, adding or step-counting, e.g. "three twelves... 12, 24, 38".
Verbal count sequence (VC) Student makes an error in their counting, e.g. "26, 27, 29, 30".
Visuospatial/kinesthetic (VK) Student makes an error relating to the physical aspect of counting, e.g. desynchronisation of verbal count and gesture, confusion over which units have already been counted, etc.

Figures 1-5 [For insertion into main text]

Figure 1: Perspective drawing of cuboid

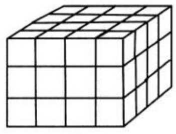


Figure 2: Isometric drawing of cuboid

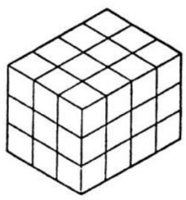


Figure 3: Colours and calculation (Leo)

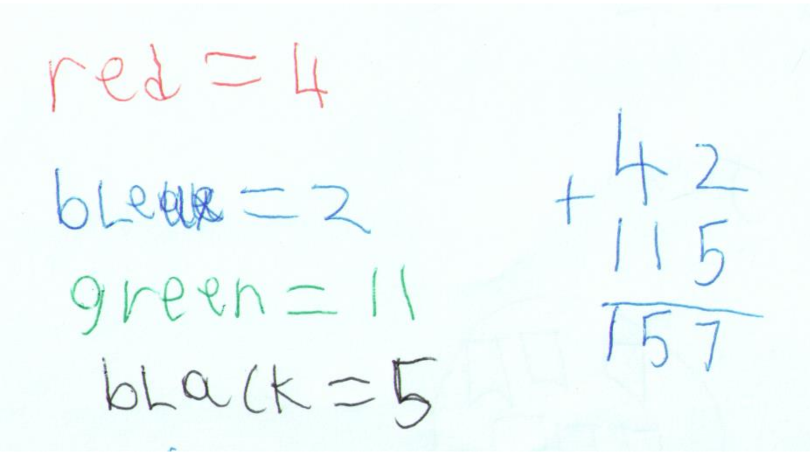


Figure 4: 'Drawers' (Leo)

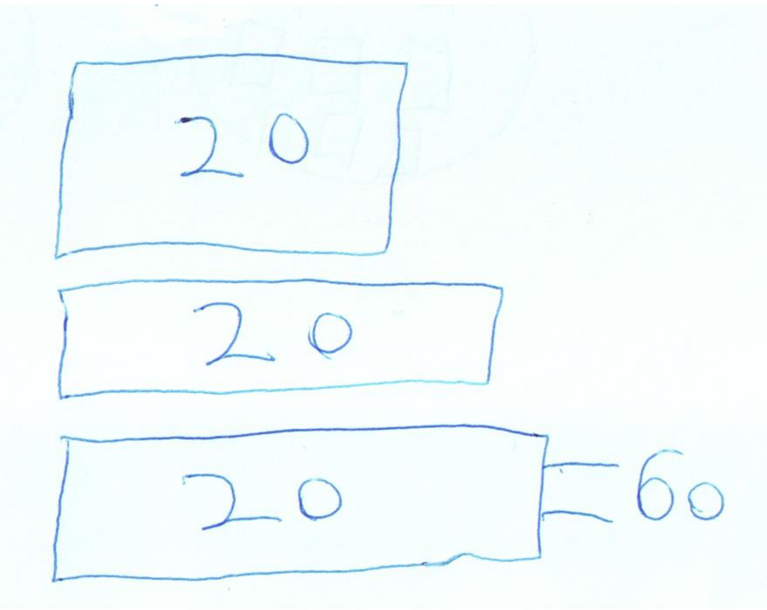
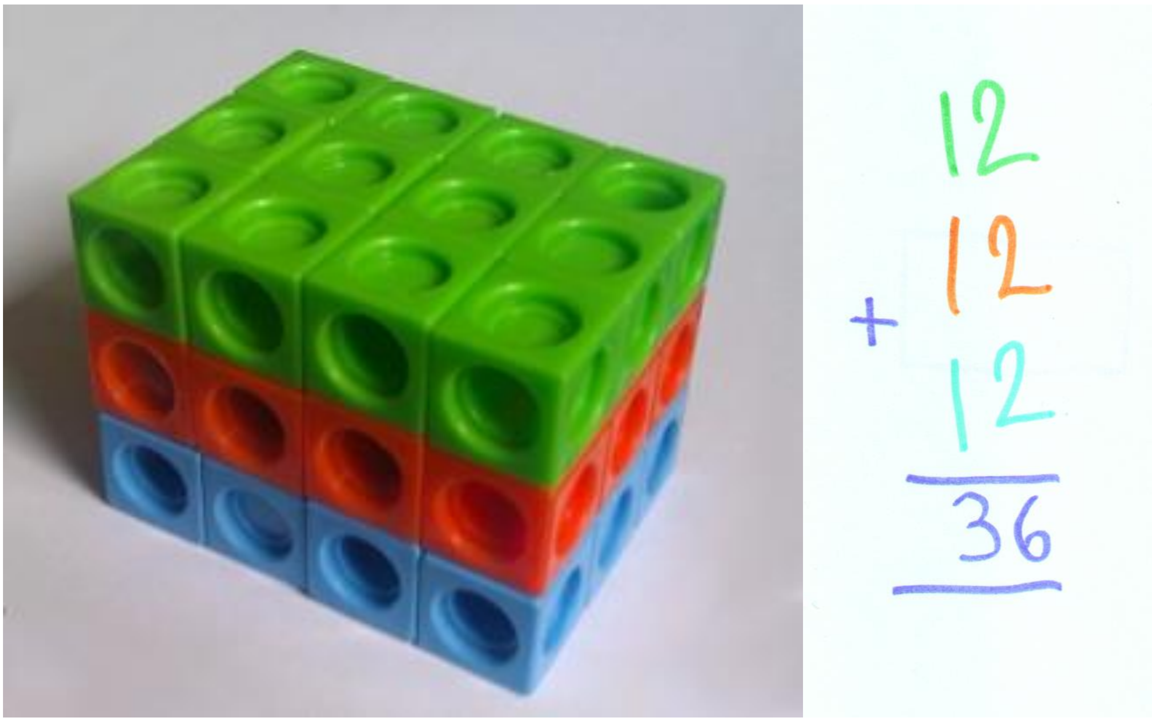


Figure 5: 3-colour block, 3-colour sum (author and Paula)



N.B. These two images should be placed side by side as a single illustration.