Visual representation in mathematics: Five case studies of dyslexic children in Key Stage 3

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Abstract

This research project is an exploration of the visual representations used by children during school mathematics, focusing on the topics of basic multiplication and division. It takes as central the theme of individual educational experience, abilities and attitudes.

The cases studied consist of five children aged 11-14, who had been diagnosed with dyslexia and for this reason sent to a special (or 'specialist') school in London. These students took part in a qualitative assessment by the author, in the form of 'problem-solving interviews', and then received a series of one-to-one tuition sessions involving work on the principles of multiplication and division, and problem-solving activities using these operations. Students used a variety of different visual representations in the form of concrete materials and/or drawings, some suggested by the author and some of their own devising. This study is a detailed documentation and analysis of this process and the representations used.
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Introduction

There is increasing concern over the number of people in the UK leaving school with poor numeric and mathematical skills. In the words of The Basic Skills Agency (BSA):

Over the past two decades in the UK there has been widespread concern among educators, employers and politicians about levels of basic numeracy among the population as a whole. In a recent international survey of adult numeracy standards, UK respondents demonstrated the lowest numeracy levels of seven participating countries (UK, Australia, Denmark, France, Japan, Netherlands and Sweden).

(BSA, 1997, cited in Mulhern, 2004, p1)

There has been an increasing amount of practitioner-led research into good practice with regard to teaching methods, for example, the Low Attainers in Mathematics project (1979-82), Children's Mathematical Frameworks (1983-5), the National Numeracy Project and Mathematics Enhancement Project (1997), and the Raising Standards in Numeracy project (ongoing). Further investigation into the way mathematical principles and procedures are assimilated and utilised both generally and by lower-attaining students is helping achieve a better understanding of how innumeracy occurs, and how it might be counteracted.

Helping those with the weakest mathematical abilities has been a passion of mine for over a decade, since my first teaching job at a London comprehensive, when I found that my 'bottom sets' were having great difficulty accessing the curriculum, and I began the unending experiment to find out 'what worked' for them. Throughout the subsequent years, working with the extremely disaffected young people in Pupil Referral Units, and the severely dyslexic and/or dyspraxic students in a special school (where I held the post of Head of Mathematics for seven years), I have constantly developed, refined, expanded and shared my teaching methods to support the learning of those students whose attainment is, for one reason or another, significantly below age-related expectations. During my teaching career I have found visual representation to be an invaluable tool in helping pupils build an understanding of
concepts and learn to solve mathematics-related problems, particularly the case when working with pupils with dyslexia. However, I observed that I seemed to make considerably more use of visual methods than my colleagues (partially because, as a mathematician who tends to automatically 'see' mathematical relationships and structures, my fascination with visualisation and representation is also a personal one).

Today, many commercially-produced visual and kinaesthetic teaching aids are available, including concrete materials, interactive computer programs, and widespread use of illustrations in text-based materials. However, many researchers have remarked that our education system still seems to deter students from visualising (Mamona-Downs & Downs, 2002), and this must be considered a serious inhibition, so there are clearly issues yet to be resolved in this area.

**Literature review**

**Dyslexia and mathematics**

It is 110 years since the publication of the first description of a dyslexic pupil (Pringle-Morgan, 1896, cited in Chinn & Ashcroft, 1998). Although historically, dyslexia has been seen in terms of difficulties relating to reading and spelling (Joffe, 1983, cited in Miles & Miles, 1992), more recent definitions of dyslexia take the form of a 'syndrome' (e.g. Miles & Miles, 1992), i.e. a pattern of signs of which some or all may be present, and these 'signs' include several cognitive weaknesses which affect the learning of mathematics. Yeo (2003) includes the following examples:

- Poor long-term verbal memory (meaning that the majority find it extremely difficult to memorise verbally-encoded facts, i.e. ‘times-tables’)
- Poor working memory (affecting calculations where several elements must be remembered and combined, or steps must be carried out in order)
- Sequencing problems (particularly affecting calculation methods which have been
taught as abstract procedures, and in some cases affecting counting and comprehension of the number system)

• Auditory perception and memory

• Directional confusion

• Poor ability to generalise (Chinn & Ashcroft (1992, cited in Yeo, 2003) report that students view maths as a "disjointed mixture of facts, rules and methods" which they may understand in isolation, but "frequently have difficulty in mastering the interrelationships".)

Clearly none of these symptoms (or their effect on leaning mathematics) is confined to dyslexic children alone, and as Dowker (2005) points out, most difficulties in arithmetic lie on a normally-distributed continuum between extremes of talent and weakness. Daniels and Anghileri (1995) warn against exaggerating the differences between children with 'ordinary' and exceptional needs (and also, importantly, warn against treating all children who have a particular deficit as educationally similar). It is also notable that the Dyslexia and Dyspraxia Support group now refer to "a continuum of specific learning difficulties". Thus, much of the pedagogical literature on children with a Specific Learning Difficulty (SpLD) such as dyslexia could (and should) also be applied to the teaching of other children experiencing difficulties with mathematics, or, for that matter, simply the teaching of basic mathematics.

Dowker (2005) takes the stance that it is of the greatest importance to consider students' individual differences in mathematics. 'Individual differences', of course, cover a host of cultural, interpersonal and intrapersonal variables, in addition to those directly SpLD-related, and they cannot all be included in a study of this size; nevertheless, it is individual mathematical experiences which are my focus, hence the selection and comparison of a small group of individual cases. As all students participating in this research had been previously diagnosed as 'dyslexic', I could expect to encounter some or all of the difficulties listed above during the course of the
project; however, it would be methodologically incorrect to make further assumptions.

As this project is both pragmatic and personal, analysing material from real teaching situations, I relied strongly on my own past teaching experience in deciphering the 'meaning' of the educational discourses. However, this then necessitated self-analysis and consideration of influences in my professional development, of which two, Dorian Yeo and Steve Chinn, stand out as foundational. In addition to their published works, I also studied with Yeo which, in providing a 'springboard' for my own thinking with regard to dyslexic students, necessarily permeated my practice, making it now difficult to critique their ideas as entirely separate from my own. However, I pick out one issue related to this project on which I differ (from both authors, and in fact, much of the 'received knowledge' of mathematics teaching). This is the issue of 'times-tables', a deficit in which Ginsburg (1997, in Chinn & Ashcroft 1998, p61) suggests is "the major feature differentiating children with and without learning difficulties". Researchers have long emphasised the need to study individual arithmetic strategies (e.g. Buswell & John, 1926, cited in Dowker, 2005), and it may be discovered, from conversation with both children and adults, that frequently, a mixture of retrieved facts and calculation strategies are used in mental arithmetic (Lefevre et al, 2003). In fact, Dowker goes as far as to describe the ability to derive unknown number facts from known ones as "perhaps one of the most crucial aspects of arithmetical reasoning" (2005, p123). Although I would never argue that the ability to instantly retrieve all 'tables' facts is not extremely useful, I do propose that it is not essential for 'success' in mathematics at any level, on the condition that the individual concerned has a set of reliable and reasonably efficient strategies in place. This stance informed my practice during the tuition phase of this project.

**Conceptual knowledge, constructivism and connectionism**

In this research I make an assumption that mathematics education should ideally lead
to conceptual understanding (or relational, to use Skemp's (1976) term) rather than merely rote (or instrumental) use of facts and algorithms. This is not a new idea (for example, Edgeworth (1798, cited in Dowker, 2005, p31) stated "A sensible boy is not satisfied with merely seeing ... a given question come out right, he insists on knowing why it is right") but is nevertheless one which has been hotly debated over the years, with trends in both directions (which I do not address here). This raises the question that if conceptual understanding is desired, how may it be transmitted, discovered or constructed?

The theoretical background for this project incorporates a strong element of constructivism, working on the principle that the learner should be active in building their own knowledge, constructing relationships and developing mathematical frameworks through assimilation and accommodation. This idea, formalised by Piaget, has threaded through the research which followed to the present day. In fact, Fuson et al (1997, p132) characterised the dialogue within the research community as being "dichotomized as (radical) constructivist vs. non(radical) constructivist, with the latter described as the representational view of mind." However, one questions the extent to which the constructivist principle is present in the belief systems of the teaching community, and, moreover, in their teaching practices (which may be inconsistent with beliefs).

This leads to another educational principle, the connectionist definition of "Mathematics as an interconnected body of ideas and reasoning processes which the teacher and the student construct together" (Swan, 2006, in press). This derives from the model for different teaching/learning styles defined by Askew et al (1997, cited in Swan, 2006) in a comparison between transmission teaching (influenced by behaviourism), discovery (strongly constructivist, in the Piagetian sense) and connectionist teaching (influenced by constructivism, but also the Vygotskian view of discussion as a tool for metacognition). In a similar fashion, Noss and Hoyles (1996,
cited in Pratt & Noss, 2002, p458) suggest "learners come to construct new mathematical knowledge by forging and reforging internal connections through the interaction of internal and external resources during activity and in reflection on it." As will be seen, this emphasis on connections influenced decisions on the structuring of concepts in the tuition I provided.

**Visual representation**

Piaget's oft-quoted phrase *'thought is internalised action'* provides an introduction to the influence of constructivism and connectionism in the classroom (although he was certainly not the first to suggest such an idea, e.g. "Our first teachers ... are our feet, hands and eyes" (Rousseau, 1762, cited in Adler, 1970, p15)), and hence, to the use of visual methods. Bruner took this simple concept encapsulated by Piaget as a starting point, and, combining experimental psychology with classroom practice, developed a theory that knowledge is "in the form of connected mental representations" (1967, cited in Daniels & Anghileri, 1995, p41), mapping out a theory for children's cognitive development through three types of representation: from *enactive* (actively manipulating concrete materials) to *iconic* (pictorial representation involving mental images) to *symbolic* (including use of language and mathematical symbols). This theme of development has had a great impact on subsequent pedagogy, particularly that relating to the teaching of those with SEN. Of particular note in this field were, among many others, Steeves (in 1979 the first to specifically suggest a multisensory approach to mathematics teaching for dyslexic students, cited in Chinn & Ashcroft, 1998), Sharma (both academic research and practical teaching guides), and, as discussed earlier, Chinn and Yeo.

Towards the end of the 20th century, advances in the fields of cognitive development and neuropsychology engendered an interest in the mental function of *visualisation*, and thus the link between internal and external visual representations
and its relationship to mathematical understanding and problem-solving. Sharma (1992, cited in Cogan, 2004, p3) asserted "Anything visualised goes straight into long-term memory", and while this particular statement seems far too extreme (an opinion based on teaching experience), the last two decades have produced a significant body of evidence for the strength of visualisation as a tool for learning.

Yeo (2003, p32) summarises the main arguments for the use of 'conceptual tools' (including by this concrete materials and 'flat' visual models) thus: "... conceptual tools help children construct, experience and directly visualise or 'see' maths relationships, concepts, structures and problems ... [they] help children sustain working memory processes in the early stages of thinking through problems in new maths domains ... they help children develop increasingly sophisticated webs of understanding." Fuson et al (1997) warn against any expectation of rapid and direct interiorisation of conceptual structures through use of concrete materials, and yet practical experience tells us that sometimes one can 'strike it lucky' and introduce a student to the representation by which a difficult concept or procedure suddenly makes sense to them. However, it is correct that this should not be the expected outcome of any teaching process.

Visual representations can of course take many forms, and a major problem in past research has been the definition of any kind of shared taxonomy. I adopt the idea that "a representation is any configuration (of characters, images, concrete objects, etc.) that can denote, symbolize, or otherwise “represent” something else" (Palmer, 1978; Kaput, 1985; Goldin, 1987, 1998; cited in Goldin, 2002, p208). It is also necessary to differentiate between: external vs. internal representations; the function of such for abstraction or contextualisation; interaction with a given representation as active or observational. In searching for a basic taxonomy for the external representations I would encounter in my own data, my starting point was that of Elia & Philippou (2004) (based on Carney & Levin, 2002), which proposes four functions
(categories) of pictures used in Mathematical Problem Solving: a) decorative, b) representational, c) organisational, and d) informational. However, while these are appropriate for the analysis of images which are presented to students as part of mathematical problems, they are less useful when applied to student-created images, which tend to be either simply representational, or combining different types. In addition, for a small-scale qualitative project such as this, there is actually little need to categorise; rather, student images may better be positioned on a continuum between 'pictures' (decorative-representational) and 'diagrams' (organisational- and/or informational-representational).

Bruner's *enactive – iconic – symbolic* progression cited earlier can clearly be applied to and within forms of mathematical representation, with diagrammatic drawings corresponding to a more mature stage of development than pictorial. Concrete representations may also vary in their level of abstraction while still using the same materials; i.e. a child able to make an object stand for a complex set (such as a bus full of people) is at a more advanced stage of usage than one for whom one cube, or one finger, can only represent one numerical unit. Thus a simple overall developmental description of visual representations would be:

(It is, of course, not suggested that children proceed through all stages in a linear fashion.)

It appears there have been no large-scale comparisons of styles of internal visualisation in this way, but it is reasonable to suppose that similar might apply, although with the obvious complication that while external representational structures are reasonably straightforwardly accessible via observation, internal representations, visual or otherwise, are not. Goldin (2002) asserts that educated adults and some children may be able to self-analyse and discuss their thinking, but that the extent to
which internal representations may adequately be described through introspection is
questionable, and must be analysed with careful, context-dependent inference. Pratt
and Noss (2002, p462) also mention difficulties in studying such a relationship, but
nevertheless indicate the need to "get a clearer picture of how specific pieces of
knowledge and the relations between them are activated by the experience of
encountering the (real or virtual) world". Hence, while such 'pictures' emerging from my
case-study students' voiced introspections are likely to be far from clear (and possibly
few and far between), they are nevertheless to be seized upon as sources of potential
insight into children's thinking.

Recent experimental and quasi-experimental work in the field of visual
approaches to teaching and learning (e.g. Booth & Thomas, 2000; Woolner, 2004) has
provided a certain amount of statistical generalisation, often in the form of 'what works
best, on average', which is of course important for curriculum planning purposes.
However, in the real world of teaching real children, it has been noted by Dowker
(2005) and Yeo (2003) among others, that there is a huge variation in 'what works' with
different individual students. This will come as no surprise to most mathematics
teachers (although from glancing at the yearly teaching programmes and government-
imposed targets of the National Numeracy Strategy Framework, one could think
otherwise). I chose to focus on this individual level, and by limiting the number of
participants, aimed to collect the richest possible data on these carefully-chosen
cases. It was expected that this data would be sufficient to provide theoretical insight,
and then, when used in combination with the statistical analysis of other research in
the field, could provide some degree of analytic generalisation.

The literature discussed above provided a theoretical background for this
project as a whole. Further sources used specifically in the planning and
operationalisation of one of the three 'Phases' of fieldwork will be discussed in the
relevant chapter(s).
Research questions

• What types of visual representation are there in the mathematical experience of the students at this school?

• How can the use of visual representation facilitate low-attaining students' understanding of mathematical principles and procedures?

• How can the use of visual representation aid low-attaining students' mathematical problem-solving?

Methodology

The project design followed a flexible, multiple-case study approach with the aim of collecting rich qualitative data on a small but varied group of students within the 11-14 age range. This age group was chosen for several reasons: I did not want too wide an age range, as the school mathematics curriculum is structured differently for Key Stages 3 and 4; pragmatically, it would be more difficult to arrange to see older students, as they would be concentrating on the GCSE course; and I felt I knew them too well (in the sense that the prior relationship might bias their behaviour with me). Another advantage of choosing younger students was that they would have joined the school recently enough for them to be able to remember their previous mathematical experiences, as the move from primary to secondary school is a time of great change (shock, even) for many children.

The project was also influenced by the ethnographic paradigm. Initially I chose to follow the approach outlined by Frank (1999) in my use of lesson observations to 'get a feel' of the students' classroom experience. However, I came to realise that the knowledge and understanding of this community gained during my previous years teaching at the school also constituted a kind of retrospective, much more 'in-depth' ethnography.

Although my research questions focus on one particular aspect of mathematical
experience, it is clear that students' use of visual representations is related in a complex and non-linear way with other variables, from which it makes little sense to divide it. Additionally, although my fieldwork centres on individual mathematical behaviour, for the most part in a one-to-one researcher/participant setting, their behaviour will nevertheless have been affected by that of other people; the participants' current and past mathematics teachers and LSAs, and their peers, most obviously. Family members will likely have played a part, which leads to more general questions about cultural factors, and to issues of race, gender, and socioeconomic status. As well as the 'permanent' or long-term factors above, there is the consideration of students' "beliefs ... motivation, confidence, anxiety and identity" (Evans, 2000, p109) in relation to mathematics. Decisions must be made on how much could reasonably be included.

I structured the project in three phases, as detailed below. The Observation phase would provide me with an idea of the type of (representational) educational environment in which the students learned mathematics, as observed in the behaviours of their teachers, LSAs and peers. I could observe, on a basic level, the students' relationships with staff and peer group, and their general demeanour in the mathematics classroom could additionally give clues to affective factors. Any such clues could then provide a starting point for further probing at a later date. In the Assessment phase, in addition to a 'diagnosis' of students' mathematical strengths and weaknesses, I would intersperse general questions about their relationship with mathematics, which again could be followed up in tutorials, once we knew each other better. In terms of choosing which personal, social (etc.) factors to include in analysis and discussion, in both the Assessment and particularly the Tuition phases, I opted to take my lead primarily from those factors which the students themselves chose to bring into the discourse.
Phase 1: Observation (2 weeks)

(all students in KS3 and some staff)

• Observation of at least one mathematics lesson for each KS3 class

• Selection of case studies

• Inspection of background documentary evidence, in the form of the case-study students’ educational records

Phase 2: Assessment (1 week)

(case-study students and some staff)

• Semi-structured ‘problem-solving interviews’ (as used by Clements & Battista (1992, cited in Booth & Thomas, 2000) involving informal pre-testing of students' basic mathematical abilities and use of pen-and-paper visual representation, including discussion of their chosen strategies for solving problems; also some general discussion about their mathematical experiences

• Brief unstructured interviews with the students' mathematics teachers and LSAs, for further background information

Phase 3: Tuition (4 weeks)

• Weekly one-to-one tutorials on a component of the mathematics curriculum in which participants have difficulty, locating as precisely as possible then addressing the problematic concepts and sub-processes, with ongoing analysis of the progress made, including students’ ability to explain a principle or procedure and their ability to solve (or make a reasonable attempt at solving) mathematical problems

Ethical issues

This project was carried out in compliance with the BERA (2004) Revised ethical guidelines for educational research. The majority of ethical issues pertain to the concept of an ethic of respect for the person ('the person' in fact representing the case-
study students, other student participants, and staff). During the fieldwork and its reporting, care was taken when making any references to students’ age, gender, ethnicity, socioeconomic status or family situation. Poor academic performance can also be linked to issues of self-esteem, with Mathematics being a particularly emotive subject; hence I needed to be constantly alert to any indications of distress in participants, and to take appropriate measures to counteract this, should it occur. This necessitated taking care to build up a trusting co-operative working relationship with all participants.

Legal issues relating to consent and privacy apply, and all names other than my own have been changed. Consent for the project was given by the Headteacher and Board of Governors, all parents were informed of my researcher status via the school newsletter, and all pupils were informed in a school assembly. Participants were given brief informal feedback at the end of the fieldwork period, and staff were given the opportunity to read the finished report.
Phase 1: Observation

Rationale

"There is no such thing as the description of a classroom ... There are an infinite variety ... and in order to judge the adequacy of [a] description it is absolutely necessary ... to know the purpose for which it has been arrived at."

(Croll, 1986, p3)

Robson (2002, p311) suggests observation as highly appropriate "in an exploratory phase, typically in an unstructured form, to seek to find out what is going on in a situation as a precursor to subsequent testing out of the insights obtained". This was my intention, with the four purposes below.

Overview of visual representations employed

The first aim was simply to "get a feel" of Mathematics lessons in the school. In particular this would include obtaining an approximate impression of the frequency and nature of the representations created by and presented to students during mathematical activity. As my focus was on the students' experiences of learning mathematics rather than the teachers' of teaching it, data collected on the former was expected to greatly outweigh the latter. Teacher actions and behaviours would be noted in passing, but transcribed in detail only when they related to the use of visual representation.

First hypotheses

Croll (1986, p8) suggests "A period of qualitative observation can provide researchers with procedures, definitions and hypotheses for an investigation". It was intended that this preliminary observation of students' classroom activity would help me become aware of potential patterns in students' interaction with representations, and suggest possible relationships between these and the student's mathematical abilities and
attitudes; I expected from this data to form some tentative hypotheses about students' use of visual representation.

Selection of cases

My particular interest being in low-attaining students, I hoped to find among these a subset of participants displaying a variety of individual abilities and characteristics. It should be remembered that the case-study students would be receiving individual tuition in arithmetic and mathematical problem-solving, and thus an ethical factor was also present, i.e. that students should be selected whom the teachers and I hoped would particularly benefit from this. (Of course, it may be suggested that all students could benefit from extra one-to-one tuition, but priority was in most cases given to those who seemed least able to access the curriculum through existing educational circumstances.) I actually expected that six would be too large a number for the depth of study I intended; however, I had to consider the possibility of sample mortality through absence from school.

Familiarisation and de-familiarisation

"Make the familiar strange and the strange familiar"


As I had previously taught at the school, I already knew a large proportion of both students and staff, and had a good 'insider's' understanding of the organisation, practices and ethos of the school community. Illustrating both negative and positive aspects of this, Wragg (1999, p15) suggests "Insiders can sometimes find it difficult to detach themselves from their own prior knowledge, beliefs, commitments and prejudices about a place they know very well and have seen every day for years. On the other hand they often understand the significance of events that might elude strangers." My relationship with the school situated me at a point which was neither
'insider' or 'outsider'; classroom observation could help me re-assess what I thought I knew.

Walker (1976, p9) comments "What you see in a school as an observer is partly a function of how the school sees you." In other words, the observer is also observed, and hence another product of my presence in lessons was to allow the students to become familiar with my role as researcher.

**Method**

**Observation typology**

There have been many attempts to classify different observational approaches (e.g. Gold, 1958, Adler & Adler, 1987, both cited in Angrosino & Mays de Pérez, 2003; Robson, 2002), the emerging pattern being essentially a continuum with the complete participant and the complete observer as extremes. A 'middle ground' approach was required, some examples of which being observer-as-participant (Gold, 1958, in Angrosino & Mays de Pérez, 2003), peripheral-member researcher (Adler & Adler 1987, ibid.), or unobtrusive observer (Robson, 2002). The latter is defined by being non-participatory in the sense of being non-reactive, and is also often unstructured and comparatively informal, which suited my purposes.

**Sample of lessons**

The school currently has 50 students in Key Stage 3, each year group 'setted' into upper and lower groups for mathematics, with a maximum of ten students per class. I intended to visit each class at least once, the lower sets preferably twice. This initially created a hurdle, as the teachers of 2mB and 3mB (the lower sets in Years 8 and 9) were wary of anything which might disrupt their classes, which were "a bit unsettled at the moment" (Ms Leigh, personal correspondence). The potential disruption of lessons was a valid ethical consideration, but my presence did not appear to be disruptive; in
fact, students' behaviour was reported by the teachers as being better than usual. The intended nine periods were observed.

**Procedure**

In classroom observation it is imperative to agree with the class teacher the observer's location in the classroom and level of interaction with students. Regarding location, we agreed I would sit near the back of the classroom whenever they were 'leading the lesson', but that I could move around the classroom when the students were working individually or in groups. Regarding interaction, all three teachers allowed me to talk to individual students during the lesson if I wished. In fact, I intended to interact very little at this stage, although it was helpful to have the option of following up any interesting observations with students.

**Recording**

Audio recording, as used in the interviews and tutorials, would not be as helpful for class lessons, particularly as my main interest was in visual aspects; hand-held video also seemed excessive, potentially disruptive, and involving many complex decisions (i.e. where to point the camera). I decided to use the traditional pen-and-paper method, supplemented by photocopying examples of visual representation from students' books or worksheets and photographing any constructed concrete representations. I initially considered some level of quantification, listing and ticking off potential events relating to my area of research (e.g. student chooses to use Base-10 blocks; question requires student to draw a diagram), but decided that a 'blank page' approach was preferable at this stage. Thus I adopted a two-column system of narrative-based notes, the preferred method for ethnographers such as Frank (1999), dividing observation notes into *note-taking* (descriptions) and *note-making* (interpretations of what is observed).
Results

Overview of visual representations employed

During the lessons a variety of visual representations of numerical concepts were used, some examples being:

• Directed numbers represented by number lines and contextualised by thermometers
• Addition and multiplication in the form of rectangle perimeters and areas
• Multiplication represented as a series of 'jumps' along a number line
• Equivalent fractions represented by 'pie-slice' diagrams
• Number sequences represented by 'dot patterns' made with counters (below)

In addition, topics were taught where visual representation is an integral part, for example, co-ordinates, gradients, etc. The speed with which students adopted the above representations, and the extent to which some of the students appeared to rely on them indicated that this was an important aspect of their learning.
**First hypotheses**

I was able to make the tentative working hypothesis that, while drawings (provided or self-created) and physical objects appear to be of considerable help to students, they frequently do not make use of these unless specifically instructed to do so. Further, some students appear to be under the impression that they should not draw during the working-out of problems, with many more thinking that not to do so is somehow 'better'.

**Selection of cases**

Six students were selected for case study, three from 2mB (Nelson, Liam and Kevin) and three from 1mB (Michael, Julie and Terence). A brief 'portrait' of each is given.

Nelson is 12, and from a Portuguese-speaking home (although according to the school Speech and Language Therapist (SaLT) his understanding of spoken English is age-appropriate). He has an SEN Statement and was also assessed by an Educational Psychologist (EP), these documents dating from 2002. At the time of these reports, he fell into the 'below average' range on the Wechsler Intelligence Scale for Children (WISC), was in the 1st centile on the arithmetic component, had SpLD of a significant order, and his "attention span [was] extremely short". Although all reports highlighted Nelson's considerable "confusion over numeracy", he was recorded as "always deny[ing] problems, saying that he found everything easy". This was interpreted by the EP as "protecting his fragile self-esteem". Interestingly, in Nelson's current Individual Education Plan (IEP) he himself listed mathematics as one of his weaknesses, which, if one accepts the EP's analysis, could be seen as a positive sign of acceptance of his problems and willingness to work on them.

Liam is 14, and placed one academic year below his chronological age. He has an SEN Statement dating from 2004, in which his WISC score placed him in the 'below average' range, with "difficulty translating his thoughts into words", "visual perceptual
problems", and "a weak auditory memory ... associated with a difficulty in working with mental arithmetic". He has SpLD, but "cognitive assessments show[ed] that his non-verbal reasoning skills" were in the average range. It was recommended he be taught "the use of strategies ... to compensate for difficulties with working memory." Liam is also reported as having low self-esteem, and as being "confused a lot of the time". In Liam's current IEP he listed mathematics as one of his weaknesses.

Kevin, 13, was also selected as a case study. However, he is a frequent school-refuser, and went through a period of particularly poor attendance during the fieldwork stage, so could not be included.

Michael is 12, and joined the school mid-year. His educational background files were incomplete, with no previous reports or assessments, although the school SaLT was in the process of assessing him. She informed me that although he appeared to read text competently, he had significant language comprehension problems, and his current IEP targets referred to improving his "understanding [of] the relationship between words" and "ability to make deductions". Michael's mathematics teacher reported that he "finds most concepts in mathematics very difficult to comprehend", and, more informally, "Who knows what's going on in his head?" From my observations, he appeared to need almost constant attention from the class LSA.

Julie is 12. Her SEN statement (2005) reports that she was originally referred due to her anxiety and low self-esteem, but was discovered to experience "SpLD that include marked ... auditory processing difficulties". Her "grasp of mathematical concepts [was] very limited" although she had "average abilities in most areas". Multi-sensory teaching was recommended, and Julie's mathematics teacher suggested her participation in this project, stating she had "quite well-developed coping skills, maybe involving visualisation" which raised her achievement in mathematics.

Terence, 11, appeared to be one of the strongest students in his class. He had been assessed by an EP, who reported that his WISC score placed him in the centre of
the average IQ range, but there were indications that his real abilities may be considerably higher. He had some of the classic symptoms of dyslexia, although "functionally literate", and is also described as "dyscalculic" and "struggling with very basic mathematics", a surprising comment, for reasons which will become clear when I describe my interviews with him. Terence gained my immediate attention during classroom observation by the fact that he doodled and drew almost constantly, and I intended to find out if this corresponded to a heightened use of drawings and diagrams in mathematical problem-solving.

**Familiarisation and de-familiarisation**

It was extremely useful for me to simply spend some time in classrooms before embarking on the main part of my fieldwork. One particular aspect is of note: the fact that I frequently had to repress my instincts to react to students as a teacher, and to the teachers as a Head of Department. When asked for help by students, the decision had to be made each time how to respond. Had I positioned myself as a 'pure observer' these requests would have been ignored, but under these circumstances such interactions with students provided potentially useful data on the problems they were experiencing with the subject matter or question style, and the type of prompts which proved helpful to them.

Word had spread among the students that I was "writing a book about them", an explanation not quite accurate, but which they seemed to find sufficient. They were generally accepting of my presence, and appeared unbothered by my observing them.

**Discussion**

Wragg (1999, p54) states "We often interpret events as we wish to see them, not as they are... Good qualitative analysis of classroom behaviour involves rigorous scrutiny of ... barriers to accurate perception." In relation to observed teaching methods, I had
to stop myself being distracted by involuntary comparisons with how I would have taught the same material, but on the other hand, these comparisons were not without a value of their own. In relation to the effect of the observer on the setting, my impression of the classes was that they were behaving 'normally', or at least, how I would expect. However, student discipline is not too closely related to my main field of enquiry; indeed, it was actually in my interests for students to be more focused, as this increased the probability of my observing their use of representation as they worked.

Regarding the students selected for case study, it may be noted that I included age, gender and educational background in my descriptions but not information relating to race, socioeconomic class or family situation (e.g. single-parent status). Analysis using these factors (and, in fact, age and gender) is outside the scope of this study; however, I will note that the case-study group included a great spread of representativeness in this area.
Phase 2: Assessment

"[A] very important part of the job of a teacher ... is to guide the child towards tasks where he will be able objectively to do well, but not too easily, not without putting forth some effort, not without difficulties to be mastered, errors to be overcome, creative solutions to be found. This means assessing his skills with sensitivity and accuracy, understanding the levels of his confidence and energy, and responding to his errors in helpful ways."

(Donaldson, 1978, p114)

Rationale

This was my first contact on a one-to-one basis with the students selected as case studies, and I needed to assess their basic mathematical skills. A traditional way would be to give them a formal pre-test; however, that tactic would be inappropriate here, the benefits being outweighed by disadvantageous side-effects. Mathematics is not a popular subject for many people, and particularly so for students who struggle with it on a daily basis; similarly, formal testing is a stressful and unpleasant experience for many people, and can be a real source of dread for students with learning difficulties. Hence, although I did not yet know my participants' individual feelings towards the subject, it seemed likely that singling them out for an 'extra maths test' would not be conducive to the trusting, co-operative working relationship I hoped to form. 'Problem-solving interviews' gave the opportunity for flexible two-way interaction with each participant and so enabling discussion of the mathematical strategies used, rather than having to attempt to deduce these from the minimal 'working out' shown by most students in non-interactive assessments.

The second benefit to using this particular style of interaction relates to the nature of the researcher-participant relationship and to the other type of information I wished to obtain from the students, namely their feelings and attitudes towards mathematics. It is to be remembered that the participants were pre-adolescents being requested to talk to an adult in an autobiographical and self-analytical way about their
mathematical experiences. In pilot interviews I had not included a problem-solving element, and despite the fact that I had chosen students with whom I was familiar and whom I knew to be talkative, conversation had been stilted and, at times, awkward. Hence, my decision to combine personal questioning with mathematical problem-solving meant both that 'chatting' could provide a welcome intellectual break between 'doing problems', and that moving onto the next problem could provide a smooth exit strategy for when the momentum of the conversation came to a standstill.

As noted previously, any conclusions on students' use of visual representation make little sense without also taking into consideration their basic mathematical abilities, and are contextualised further by their general feelings and attitudes towards mathematics. Data collected from these problem-solving interviews is organised for analysis in this threefold manner.

Method

Instruments

The structure of the problem-solving aspect was determined by three worksheets, each containing a multi-part problem. Context and questions were given in the form of text, which was kept as brief as possible, and worded in a way that it could easily be read to the student in short chunks. It was important to choose contexts very carefully, as on the one hand, they can increase 'involvement' (Evans, 2000) and induce 'metaphoric thinking' (Winter, 1992, cited in previous), but on the other, can confuse children and induce the assumption of false premises (Donaldson, 1978). Booth & Thomas's (2000) experimental work on children's problem-solving provided ideas for question contexts, some of which were also adopted by Elia & Philippou (2004), although Booth & Thomas's analysis was mainly quantitative, and in both of these cases the participants were observed solving problems (as opposed to my interactive procedure). I adapted the questions to better suit my particular group of students, on
which further explanatory notes are in Appendix 3, with completed worksheets.

Procedure

A laptop was used for audio recording, placed within clear view, and I transcribed the recordings the same week. Exact timings were not included, but the original audio files were saved for further analysis if needed. Sessions were intended to last 30 minutes, but there was flexibility in both directions.

I chose a relatively unstructured format for the interviews. In attempting to elicit children's educational experiences I was influenced by the feminist principle of "giving voice to [the] hitherto voiceless" (Hollway & Jefferson, 2000, p3), although Riessman (1993, cited in previous, p3) does claim, "We cannot give voice since we do not have direct access to another's experience. We deal with ambiguous representations of it - talk, text, interaction and interpretation." One of the main feminist criticisms of traditional interviewing is the inequality in power relations between researcher and subject, and in conducting research involving minors such inequality is inevitable. However, I made several decisions regarding my language and behaviour with the intention of fostering a collaborative atmosphere. These were:

- Sit at right-angles to the student, to decrease potential discomfort arising from direct face-on observation.
- Use informal speech patterns (e.g. incomplete sentences, 'yeah' rather than 'yes') to give a feeling of naturalism.
- Occasionally share comments about my own experiences where appropriate (e.g. "I had a hard time learning my tables too"), positioning myself openly as an individual with my own history – some of which I was prepared to share with them – rather than an inscrutable researcher. As Fontana & Frey (2003, p64) state, "Each interview context is one of interaction and relation; the result is ... a product of this social dynamic".
Analysis

Assessment of numeracy levels were derived most obviously by students' ability to solve the arithmetical problems, the amount of help they needed to do so, the strategies chosen and their ability to explain such strategies. Conversation about mathematics also provided oblique indications of their ability.

Analysis of participants' use of images rested on their ability and inclination to produce their own visual representations of an arithmetical situation, to continue a representation I had begun, and the way they used these. Another factor of interest was the degree of abstraction in their representations, i.e. 'pictures' of how the situation would look to an observer, or more symbolic 'diagrams'.

Data on participants' attitudes to mathematics were derived from the brief sections of 'interview' before and between arithmetic problems. My analysis of these was essentially pragmatic, but was influenced by conversational and interpretative phenomenological analysis, the aim being "to explore in detail the participant's view of the topic under investigation" (Smith et al, 1999, p218).

On several occasions in the following analyses, I describe children as 'unable' to make an attempt on a problem; this is a considered decision. Although it may be remarked that such inaction could be a result of participant choice not to act, rather than of inability, on reviewing their overall input and perceived attitude during my time with them, I conclude that this proposition may reasonably be discarded, as I saw no indications they were not content to take part and keen to demonstrate what they could do.

Analyses are presented in the order in which the interviews were conducted.
Levels of numeracy

Like many a dyslexic student, one of the first things Julie told me was that times-tables were hard. This is a very ambiguous comment, and can mean anything from a complete lack of understanding of the multiplication operation, to not being able to instantly retrieve all single-digit multiplication facts. In Julie's case, the fact that she could successfully step-count in fives (as demonstrated in Groups) indicates that she has at least a basic conception of numerical structures. On being presented with the first worksheet, Stairs, she answered quickly but incorrectly; however, on being helped to visualise the scenario, she was able to answer correctly. Her original answer of eleven for Q1 implies that she had realised she needed to add the 'up five' and 'up six', but was not sure what to do with the 'down three' in between. Her answer of twelve for both parts of Q2 implies that she had understood it to be asking the length of the staircase from bottom to top (or vice versa), and had rejected the additional information as surplus.

On Rose Bushes Q1, Julie could manage each stage of the working when it was broken down for her, but surprisingly, completed Q2 very quickly. This could indicate either that she had understood well the suggested method and was able to reapply it, or possibly that she was simply replicating the entire of Q1, complete with answer. (This may seem like a poor choice of numbers on my part, but I was interested to see if any of the participants would comment on the fact that the two gave the same answer.) Julie could answer Groups Q1 unaided, and Q2 with support, indicating at least a basic, perhaps informal, conception of the 'grouping' form of division.

Use of images

Although Julie affirmed my enquiry as to whether our joint drawing for Stairs Q1 was
helpful, she did not attempt a similar type of solution for Q2 until I began it for her. This was also the case with *Rose Bushes*. On *Groups* Q1, however, she immediately created a simple, efficient representation of twenty (tally marks), divided them into fives (by ringing groups) and partitioned off two from each group. This appeared to be a method she was very comfortable with, so rather than simply require her to do the same again with a larger number, for Q2 I suggested an alternative: having a symbol to stand for each group of five. Julie accepted this representation, and applied her 'ringing' method to it. While for the most part Julie did not show initiative in creating images to solve problems, she was, however, able to understand, complete and replicate suggested representations.

**Attitudes to mathematics**

Although Julie told me that "some of the stuff is quite hard", she stated that she "got on with maths fine", and that it was getting better. As she seemed a well-mannered child, I wondered if she was simply being polite; however, at no point did she give indications of negative emotions toward the subject. There were topics that she liked (ones involving drawing shapes), and she assessed herself as managing "quite well" compared to the others in her class. This confirms Julie's teacher's description, and is also particularly notable (and encouraging) coming from a child reported to have issues of anxiety and self-esteem.

**Michael**

**Levels of numeracy**

Michael's reading aloud was unusual in its lack of expression, with minimal variation in tone or speed. This was consistent with comments made by the school SaLT, that his reading of words is "quite mechanical", and should not be taken necessarily to imply understanding. He extracted relevant data from word problems (e.g. "Five steps, three
steps and six steps"), but did not seem able to comprehend the relationship between them without significant help. He appeared confused by the changes in direction (up/addition and down/subtraction) despite the fact that the numbers involved were very small. When using the staircase visualisation he did not attach numbers to each stair and count up and down, but completed the physical movements separately and only counted the number of stairs at the very end. This raised the suspicion that he may not be able either to 'count on' from a number other than one, or to count backwards.

On reading Rose Bushes, Michael was unable to begin, and needed much help in setting out the problem. The numerical aspect of these involved only counting, but even this caused some difficulty as, although he could count the number of bushes he had drawn, he was unable to apply the same skill to the 'pacing out' of the 'path' in metres. Also of note is the fact that Michael wrote 'cm$^2$' after both answers. It is probable his class had recently been studying Area, and he had been told repeatedly to write this after each answer. He did not do this on the other worksheets, which suggests that he may have made some link between questions about length (here) and about area (in class).

Michael needed the Groups questions to be broken down into small chunks, and much direction. During his drawing of twenty 'men', he lost count twice, displaying a difficulty in counting when there were time lapses for drawing. This may indicate short-term memory difficulties in retaining the last number counted, or may indicate that counting for him is essentially a verbally-based recitation of numerlogs (easier to perform at speed) with little thought for magnitudes. This is further indicated by his willingness, in the second question, to draw 100 men. Although it may be argued that he simply did not mind drawing so many, it strongly suggests to me that he does not have a clear idea of how large a quantity the number 100 actually represents and so was unable to make any kind of estimate of the time that it would take to represent
this way. Persuaded to draw small circles rather than whole people, he managed to count successfully to 100, suggesting again that speed is a factor in his ability to count without error. After the grouping of the 'people' into 'girls' and 'boys', he had to count them, and on several occasions lost count because his pointing and naming became desynchronised (something more commonly seen in preschoolers, according to Gelman & Gallistel, 1978). In fact, this also happened when I did the pointing for him, evidence that Michael cannot vary his counting speed to order. When he lost his place at around 60, I suggested he simply carried on from 60, rather than starting at the beginning, as he had done on previous occasions; this was clearly a problem, but with my prompting he managed to count from 62 to 75, significantly speeding up as he refound the pattern.

**Use of images**

The first striking feature of Michael's images was their realistic nature. Unlike other participants, whose drawings generally showed the minimum of information, his took considerably longer to produce and contained more detail, much of which could be considered superfluous (but clearly was not, to him). On *Stairs*, Julie and Nelson represented movement up and down the stairs by pointing (and probably did not need the stick man I drew); Michael drew more men at each stage of the calculation. In *Groups*, the other participants represented the students by 'tally' marks; Michael drew twenty men, all complete with arms and legs (and would have done the same to represent 100 if I had not suggested an alternative). This indicates to me that although his ability to visualise is strong, and could be a very important factor in his mathematical development, his use of the visual sense is very literal, and transition from pictorial to symbolic imagery will not come naturally.

One aspect already mentioned is Michael's concept of the magnitudes of numbers. In *Stairs* Q2 he started drawing his staircase too high up the page, and did
not realise this until he actually ran out of space; my interpretation is that he does not have a strong concept of what sort of size the number twelve is, and was simply drawing steps and reciting numerlogs until he found himself saying the word 'twelve', at which point he would stop. I also suggest that he has very little concept of the decomposition of numbers into factors, as demonstrated by his linear representation of 100 (in Groups); in my experience even children with extremely weak multiplication and division skills tend to know that 100 is made of ten tens, and represent it as ten rows of ten. In addition, I make the tentative hypothesis that this tendency to linear representation indicates a poor understanding of the decimal structure of our number system.

Michael did not show any inclination to create visual representations of the problems unless I directly suggested it. This may be simply because he has little experience of doing so, as he appeared to quickly adopt and work with my suggested images. In Groups he needed the least suggestion, and (in Q1) created his own independent representation; this may have been because by the third worksheet he was becoming used to the idea of depicting situations, or perhaps because visualising a class of children came more naturally than the other contexts. Interestingly, Michael seemed disinclined to alter images that I (or he) had drawn, for example, to add more steps to a staircase, or cross out an extra bush.

Before starting the problems, I asked "Do you ever find that pictures or drawings can help you in Maths?" and Michael answered no. Later I asked "Did you find it helped you, to picture what was going on?" and he replied that he didn't know. Although there are clearly ethical issues about assuming that the researcher knows better than the subject, in this case I must disagree with Michael and conclude from his performance that 'picturing what was going on' definitely did help him.
Attitudes to mathematics

Michael's first comment on mathematics was that it was "a bit boring", and "not really my favourite subject". This is unusual, as in my experience it is generally the students who find the work unchallenging that describe it as "boring", rather than those who find it very difficult. On enquiring which bits he liked, he answered "easy ones" (although then could not think of an example of anything easy!) I suggest that his boredom might stem from the fact that mathematics has become an experience of being instructed to do many tiny task segments, broken down so much that they have lost any overall coherence. Michael also answered affirmatively my enquiry about whether he gets "stressed out" or "frustrated" by mathematics (confirmed by his outburst of annoyance when working on one of the problems).

Apart from several instances of "don't know", and "don't get it" in Michael's speech, there were also very long pauses where he appeared confused about what I wanted him to do, and many requests for questions or instructions to be repeated one or more times. Michael's manner gave the impression of one quite accustomed, even resigned, to being confused by mathematics, and without any real expectation of it making sense.

I expected feelings of failure to be a strong theme among students with weak mathematical skills, accustomed as they must be to others in their class grasping new concepts more quickly, working faster, needing less help and obtaining better marks in assessments. Michael's self-assessment of his ability, was "I do ok" compared to others in his class, which although inaccurate, is probably fortunate for his self-esteem. Although in a contrasting statement Michael sadly comments "I keep forgetting ... always do", it is generally a positive, if surprising, outcome that he does not have more negative emotion towards mathematics.
Liam

Levels of numeracy

Although Liam had a poor opinion of his own ability, his answers to my questions were some of the quickest of the group. Where he had problems, these resulted from confusion over the meaning of the question or failing to hold the necessary facts in short-term memory, rather than any inability to carry out the calculations. He was capable of all the mental arithmetic required for Groups, although it may prove important that his language did not always match the numerical procedure he was engaging in (for example, saying "five groups" while constructing five per group and vice versa.) Although Liam commented "This is hard!" during his mental calculation of \((100 – 25)\), he did not actually take long to work it out.

Use of images

Liam made little use of external visual representations during these tasks (apart from Rose Bushes, where, as expected, everyone required a diagram), although there are indications that he may be using internal ones. For example, when Liam became confused on Stairs Q2, I gestured with my hand to indicate movement up and down stairs, and this proved enough of a prompt for him. The exception was Groups Q1 where, once he had understood the question, he independently created his own representation using ringed tally marks. Although of a similar type to Julie's, Liam's groups of five were in the ‘dice’ formation where Julie's were linear, which may indicate a significant difference in their general conception of number. Also, where Julie made twenty marks and then grouped them, Liam drew circles first and then placed five marks in each; this could mean they have different ways of experiencing the phenomenon of division.
Attitudes to mathematics

Liam was the only one of the group whose manner indicated some resentment, perhaps defensiveness. His first comment, "Boring", was uttered in quite a combative manner, as were other phrases like "What d'you mean?" later. He confirmed that mathematics "stressed [him] out", but appeared to become more comfortable with me after a few minutes, and did not seem to mind my questions – or at least, preferred them to being in the class mathematics lesson. He described himself as having "always been crap at maths", and compared himself unfavourably with his peers, apart from (correctly) identifying Nelson and Kevin as being of similar overall ability. Although Liam's attitude to mathematics could generally be described as negative, he responded well to praise and encouragement, and actually grinned when I echoed his words at the end: "That was excellent - I certainly wouldn't say you were crap at maths!"

Nelson

Levels of numeracy

Nelson is the only one of the group that I had previously taught, and he made several references to this. He stated that he found the subject "a tiny bit better" (easier) than last year, but that "times" (multiplication) was "really confusing" and that "he found tests "really hard". All of this seems very believable, although, judging by his responses in this assessment, I think he may have underestimated his progress.

Nelson was able to answer the Stairs questions once they were broken down into components, although his first answers were incorrect. Likely reasons for this are that he could extract the numbers from the text but not their context, and/or simple carelessness about counting.

On Groups, Nelson demonstrated a sound understanding of the 'grouping'
division concept, the ability to step-count in fives and tens, and fairly efficient use of known facts (10 x 10 = 100; 10 = 5 + 5). His methods were slow, but he seemed confident with them. There was one anomaly, which was his almost immediate correct answer to Q3. When asked his method, he repeated the two necessary pieces of information given, but I was unable to determine if this actually indicated, in effect, an informal setting-up of the two equations, or just an echo of my words, accompanied by a good estimate.

**Use of images**

Nelson appeared to find my suggested visual representation for *Stairs* Q1 very helpful, and replicated it by himself for Q2. This pattern was repeated on *Rose Bushes*, and here it is notable that his 'bushes' are quite pictorial in Q1, but marked simply as 'notches' in Q2, indicating a tendency to move toward a more symbolic style of representation.

Nelson used tally marks in rectangular arrays for *Groups* Q1-2, implicitly demonstrating an ability to decompose numbers. However, there is a fascinating feature: in both questions, he specifically chose (as he confirmed afterwards) to set out his marks in fives, or doubled to make tens, but then did not make full use of this feature when performing the grouping operation, departing from his pre-prepared pattern and so making it more difficult again!

When discussing use of images with Nelson, his definite agreements that images in books could be useful, as could drawing his own (sometimes without being told), may be taken at face value rather than polite agreement, as Nelson has previously shown himself quite willing, even keen, to disagree with teachers!

**Attitudes to mathematics**

When I taught Nelson previously, I would have described his attitude toward the
subject as very negative; however, in retrospect I am inclined to believe that the
unhappiness, anger and confrontational behaviour he used to display were actually
more generally-directed, and partly a function of problems in his personal life, plus
insecurity over changing schools. Fortunately, his situation seems to have improved,
and he is now presenting a calmer and more positive attitude. He described certain
mathematical game activities as "fun", referring both to the current academic year and,
to my surprise, the previous one. When I enquired, he affirmed he still became
"stressed out" but less often, although still feared tests "'cause you're scared that you
might get it wrong".

Terence

Levels of numeracy

It was immediately clear that Terence's ability level was above the other participants'.
He solved the Stairs problems in a matter of a few seconds, and although he initially
made an error on the first question, it is telling that he self-corrected this. It should be
noted that many children of this age do not necessarily think to check their
calculations. Terence's explanation of Q1, "... added it ... minused it" shows that he had
translated the literal scenario into a symbolic one. He also chose to elaborate on his
mental arithmetic strategy, in effect that rather than use a straightforward counting-
based technique he decomposed numbers to utilise known number bonds, a classic
"grasshopper" strategy as opposed to Michael's "inchworm" (terms derived from Bath

Terence was very quick to answer the first Rose Bushes question, but by
assuming that this one could be as easily translated into a numerical calculation as the
previous, made the error of simply calculating 5/1 + 2. Although the numerical
component of this was trivial for Terence, he did have some difficulty in conceptualising
the given situation, and showed signs of rigidity in insisting on his original calculated
answer even when the representation disagreed. He also seemed unconvinced by my
demonstration, and went on to make exactly the same error in Q2. This is a similar
result to that reported by Lesh et al (1983, cited in Yetkin, 2003, p1), "... students who
obtain incorrect answers for their written calculation are often able to find the correct
answer by using concrete materials. However, when they are confronted with their
written work, about half of these students kept their incorrect answer."

In *Groups*, Terence demonstrated the ability to multiply and divide by five, and
in his explanation indicated further use of decomposition strategies (i.e. \(75 = 50 + 25\)),
the use of known facts \((50 = 10 \times 5\) and \(25 = 5 \times 5\)), and the skill of holding and
combining partial answers \((10 + 5)\) in short-term memory.

**Use of images**

As Terence appeared quite keen and able to talk about his mental arithmetic, I asked
him if he "saw" or "visualised" what he was doing. His answer "I just kind of see stuff,
what, how much" does appear to indicate internal visual representation. However, it is
surprising, and perhaps even contradictory, that although he was keen to use diagrams
in *Rose Bushes*, they seemed of limited help to him. This could mean that his
representational style is more suited to working with discrete quantities than
continuous, spatial forms.

Images and drawing are clearly important to Terence; however, he seemed to
believe that the use of external visual representation was not appropriate in
mathematics, indicated by his surprised "Oh, am I allowed to draw diagrams?" and
resentful "I'm not allowed to draw in maths". I guessed that he had been told not to
doodle in his exercise book and had interpreted the ban on irrelevant drawings to
include any images which were not directly requested by teacher or textbook. I asked if
he thought it would help him to be able to draw while doing questions (answered with a
strong affirmative) and offered to speak with his teacher about it. This appeared to
have a very positive effect, although I wondered if he had grasped my distinction between helpful and irrelevant drawings or was just keen to have the doodling ban lifted. Terence's opinion of the visual representations provided in textbooks was not a high one. He thought they were "not actually there to help", and that perhaps the people writing the books drew pictures because they were bored!

**Attitudes to mathematics**

Terence appeared to have by far the greatest confidence of all participants, as his answers sounded certain and were not spoken with questioning intonation or a look to me for confirmation (as many of the others' were). He gave the impression of someone used to getting the right answer, and took pride in explaining how he had done so. His "I don't understand" after giving his answers for *Rose Bushes* seemed to display shock at being wrong.

Terence's current attitude to mathematics is strongly affected by his relationship with his teacher. He cited her as the reason he didn't like the subject, more than once repeating "she's really mean", in contrast to his teacher at primary school, who "was nicer, so ... it's really different". He was able, though, to dissociate the subject from the personal when specifically asked, stating "I don't mind it really". Terence then made the strong statement "I hate algebra." Although he assessed himself as able to do all the work, he also remarked that he didn't understand it. Rather than being a contradiction, I take this to be a comment of unexpected perspicacity on the instrumental/relational nature of mathematical understanding (Skemp, 1976), in that he could remember and apply the procedures to produce correct work, but was uncomfortable that he did not know the principles by which these procedures worked. It is notable that in his mind the fault lay entirely with the teacher ("I don't understand what I'm doing because she doesn't really explain it ... And then when she does, she doesn't explain it very well.")
Discussion

These problem-solving interviews provided a great deal of data on all three areas of interest, while allowing me to form individual working relationships with the case-study students. Even at this stage it was possible to use the data collected so far, 'triangulating' my observations of their actions with their own descriptions, those of their teachers, and historical information from their files, so building up an understanding of each child's relationship with mathematics. I briefly mention here some epistemological issues that emerged during this phase.

In comparing the degree of abstraction in different participants' representations, it is important to remember that this may depend not only on their ability to abstract a mathematical situation, but on their level of artistic aptitude and interest (i.e. a child who draws a 'lollipop' to stand for a rose bush may actually be trying, unsuccessfully, to draw a realistic picture; a child who draws a detailed picture may well be able to answer the question symbolically but just enjoy drawing). Nevertheless, I decided that for analysis of this feature to take place, I would assume that as a very general rule, people will draw the minimum of detail necessary to enable them to work out the answer to a given question.

It was an important realisation that, while my visual analysis was not derived from any formal scheme, it was in fact supported by the prior experience of inspecting hundreds of children's exercise books and the representations therein, forming conclusions about what they signified in terms of understanding and individual learning style. Additional support, however, was found in both pedagogical texts and the literature of experimental psychology which involved in-depth analysis of children's mathematical behaviours; this includes among others Piaget (1965) as the original model, Gelman & Gallistel (1978), Johnson et al (1989) and Nunes & Bryant (1996). Although not dealing directly with imagery, their insights helped make certain phenomena comprehensible; for example, Nunes & Bryant (1996) remark "One reason
for children succeeding in one task but not in the other is that, as far as the child is concerned, the situations are mathematically different and call on different logical invariants, despite the fact that on the surface they appear to most adults to be mathematically equivalent." This could apply, for example, to Stairs, and explain why, having worked through Q1 (with or without help), participants did not necessarily think to apply a similar representation to Q2. Gelman et al's various work, although involving much younger children, discussed errors observed in Michael's counting, and while I do not in general accept their principles-first model of numerical development, their identification of the component principles in counting nevertheless enabled some of my earlier conjectures regarding Michael's understanding.

As expected, the conversational element of the interviews did not generally flow freely. There were no indications that participants were unwilling to talk to me or deliberately concealing information, but I had a strong sense that at times they were simply not sure what I was asking them (or why). As an education professional one sometimes forgets that even highly-educated adults may not be used to consciously thinking about their learning processes, and to ask children to make comparisons between methods (i.e. verbal and visual) is expecting much. This does not mean that they should not be asked, but that their answers to such questions should be treated with caution.

Linked to the above issue is that of language. Hollway & Jefferson (2000, p8) point out, "It is a basic assumption in much social-science research that if the words used are the same, and if they are communicated in the same manner, they will mean the same thing to numerous people in a sample." Although clearly one cannot assume any such thing in the case of a conversation between a researcher and a child, one can at least encourage them to say if they do not understand, and be alert to any indications they have interpreted the question other than was intended. Again, in both my choice of language and my assessment of whether their responses 'matched up'
with my questions or elicitors, I relied on a sense of judgement formed by experience, i.e. thousands of hours spent in the company of children like these, and also noted responses which I could later follow up in the tuition sessions.
Phase 3: Tuition

After consultation with the school's mathematics department, it was decided I should focus on basic division. All students were reported as experiencing difficulty with aspects of this operation, and a flexible tuition plan could be created, which would allow for differences in ability and speed of learning. However, as the operations of multiplication and division are inextricably linked as inverses, in my view, students’ conceptions of both must be addressed.

Multiplication and division

"A common view of multiplication and division is that these are simply different arithmetic operations which children should be taught after they have learned addition and subtraction. According to this view, there need be no major change in children's reasoning in order for them to learn how and when to carry out multiplication and division. This view was challenged by Piaget and his colleagues (see, for example, Piaget, Grize, Szeminska and Bangh, 1977), who suggested that understanding multiplication and division represents a significant qualitative change in children's thinking."

(Nunes & Bryant, 1996, p142)

In additive reasoning situations, numbers represent fixed-size sets (of objects), which may be joined or separated, whereas multiplicative reasoning includes the concepts of numbers as multipliers or divisors as well as set sizes. Although there are "certainly links between additive and multiplicative reasoning, and the actual calculation of multiplication and division sums can be done through repeated addition and subtraction" (ibid.), it is unhelpfully reductionist to define them only in this way.

The teaching of these operations has inspired much debate. "In many traditional teaching approaches ... division is introduced once multiplication has been covered and children ... are expected to use a known multiplication fact to solve a division question" (Yeo, 2003, p332). In other cases the debate has centred on whether division should be introduced as one or "as two kinds of operation, partitive and quotative" (Marton & Neuman, 1996, p319). To a teacher of children with SEN it
seems quite ridiculous to suggest that children should learn an abstract 'rule' which relies on retrieval of facts (which may or may not be adequately memorised, for reasons previously discussed), and Yeo (2003) suggests that as many of the underlying structures and methods are shared between multiplication and division, children should learn about the two concepts at the same time, with which I am in agreement. Not being in a position to introduce these concepts to students for the first time, and as a proponent of 'connectionist' teaching, I decided to make it my role to provide links between different multiplicative reasoning situations and discuss and explore the processes involved. Of influence in planning were the intervention techniques used in components of the Numeracy Recovery programme (Dowker, 2005; details in Appendix 4). The scenarios I had used in the Assessment phase had generally worked well, although were a little contrived. For this phase, I created similar scenarios, but with one main alteration: rather than have 'a child', 'a man' or 'a teacher' as subject, I phrased scenarios, as far as possible, in second person, i.e. "You are in a Food Tech lesson...". I am aware of Adda's (1986, cited in Evans, 2000) critique of the artificiality and unpredictable 'affective charge' of 'word problems', and did my best both to choose situations likely to be familiar to all students and to be observant for the possibility of distress through unforeseen negative associations with a scenario.

The multiple possible interpretations of mathematical symbols, so attractive to mathematicians, can create confusion in children's minds. Anghileri (1985) asked teachers to read out the expression '3 x 4', and obtained a variety of responses; two possibilities are 'three lots of four' and 'three multiplied by four', which, while they may give the same answer, provoke different visualisations. While some pedagogical texts stress that consistency in the classroom is highly important (e.g. Daniels & Anghileri, 1995), the fact is that students will probably have already had to cope with such interchangeable terminologies, and hence, the issue should be confronted directly and discussed. This particular instance is also where the 'rectangular area' model of
multiplication/division comes into its own, with a simple array being manipulated to demonstrate not only the different calculations, by why they are quantitatively equivalent.

**Tuition**

I decided not to impose too strict a tuition plan, but allowed myself to be guided by the questions, actions and reactions expressed by students. Nevertheless, there were certain key concepts which I intended to cover:

- Numbers may be represented visually by a set of objects (physical, drawn, or imagined), and these sets may then be manipulated to show mathematical operations and their results

- Sets of objects (i.e. numbers) may be sorted into equal groups of a given size (division: quotative or *grouping* model)

- They may also be sorted into a given number of equal groups (division: partitive or *sharing* model)

- A set of equal groups of objects may be combined to give a total number of objects (multiplication)

- There is a standard symbolic notation for these operations

- The number of groups and the number in each group may be reversed without affecting the total number of objects (commutative principle)

If I judged participants to have a reasonable understanding of the above concepts, I would introduce 'remainders', and then possibly fractions. I also kept numbers fairly small, writing to a minimum, and avoided formal calculation algorithms unless specifically referred to by students.

I designed a selection of contextualising scenarios to provoke certain lines of thinking, and construction of the above concepts. While the scenarios themselves remained constant, different numbers and wordings could be used, appropriate for
each individual. This flexibility also meant that a single scenario could incorporate and interconnect multiple types of calculation and/or concept.

‘Fairground’ involved a number of children separating into twos or threes to go on different rides. (*grouping*)

‘Baking’ involved quantities of cakes/biscuits (made in a Food Technology lesson) either being shared for eating, or being put into packets (and sold for charity).  (*sharing & grouping*)

‘Painting’ involved groups of children dividing up the task of painting a wall. (*multiplication & sharing*)

‘School trip’ involved calculating how many minibuses were needed to transport a number of children. (*grouping*)

Additional scenarios were improvised, following lines of thinking suggested by a particular student's questions or actions.

**Analysis**

All participants used both concrete and drawn representations at various times during the tuition sessions. Additionally, Michael made use of representations which were a combination of the two, Nelson and Liam used fingers, and Terence described internal visualisations. On some occasions I suggested students use either multilink cubes or pen and paper; on others I let them choose their method. Disembedded questions (i.e. without contextualising scenario) were interspersed throughout the tuition sessions. Scenarios are presented in approximately chronological order. Regarding reproduced images, my contributions are in purple, so easily identifiable.

**Disembedded**

The various tasks and questions covered all the above mathematical concepts, and responses included use of concrete, drawn and internal representations. However, one
particular representation was central: the rectangular array. Yeo (2003) suggests this as the most efficient model for demonstrating the principle of commutativity and also that of distributivity. I further suggest that it is a natural development from basic grouping work, and perhaps the clearest non-verbal way to express both of the above principles and the link between division and multiplication. Thus all participants were introduced to basic rectangular numerical representations, two examples of which are mentioned here.

Nelson particularly liked the following diagram we created together, and subsequently looked back at it several times, (I believe) to remind himself of the commutative principle and the reversibility of division and multiplication.

Julie already had a similar method of her own for working out multiplications; after successfully performing small groupings and multiplications with cubes, I asked if she could work out three times seven without them, and she produced the representation below. This has similarities to a ‘tally-mark’ representation she used in the assessment (Groups Q1), but is array-based rather than linear. This implies a link between her written methods and the cubes, which, as can be seen from my own array underneath, I then attempted to reinforce, showing that arrays can be read in two directions. Using my rectangle, she was able very quickly to tell me how many threes and sixes were in eighteen.
However, Julie declined to try drawing rectangular representations herself, preferring her own kind. In fact, I found that she could apply her arrays of marks to performing division as well as multiplication, as shown below. This seemed a reasonably reliable and efficient strategy for her (although would become less so for larger numbers), the only less positive aspect being that she needed two separate depictions to calculate sixteen divided into twos and into eights, rather than being able to use one for both.

**Fairground**

Liam and Terence were able to answer these simple questions quickly, without external representation. Nelson and Julie gave verbal answers initially, and used cubes to check, counting out the number of 'children' then putting them into groups of the
required size. Judging by Michael's blank looks, he had difficulty comprehending what was required. When I started grouping eight cubes in twos, he was able to continue the process and then count the groups. However, requesting that he then group nine 'children' in threes met with further blankness. I tried drawing, providing the following illustration:

![Illustration]

Michael then answered "1? No, two ... um... I think they're going to need three." To demonstrate, I laid three cubes on top of the stick figures, then moved them aside, three times. He nodded vigorously.

**Baking**

Julie made use of cubes in a classic way, counting out the total number then sorting them into groups, either 'dealing' them into piles (sharing) or counting out a number at a time (grouping). Surprisingly, Terence did the same, although with larger numbers and less overt counting; however, he later said he would normally do these in his head, but was "tired today". He agreed that using concretes in this way was a good "back-up plan" if he "forgot" his usual methods.

Michael showed confusion as previously, even when offered cubes. I drew 'plates' for the five imaginary children and worked through sharing out ten 'cupcakes', then asked him to share fifteen in the same way. He guessed they would get five each, and placed five cubes on the first, second and third plates, only then stating "there's not enough".
He then tried putting four on each plate, looking at it for a few seconds, before deciding "nope".

On both occasions he did not try to adjust his current sharing model, but removed all cubes and started again. On all attempts (including the correct one), he did not show any sign of being able to tell whether it would 'work' or not until he had finished. This is reminiscent of behaviour reported in his assessment, where he could not predict in advance how much space his representations of numbers would take up. On that occasion I suggested this may have been due to counting by verbal recitation only, but as on this occasion the numerosities were expressed visually rather than verbally, a perceptual difficulty may be present involving numerosities. However, Michael’s ability in representing this scenario was not consistent; he found grouping questions easier, working considerably faster and with less help, although he still needed to count out all...
cubes before deciding if a group size would 'work'.

Liam used his fingers to work out ten and fifteen shared between five, but this method failed him when applied to 24 shared between four. He tried representing 24 by drawing 24 dots; however, he subsequently added extra dots beside each (likely confusing the question with multiplication). I suggested an array format for the dots, grouping them while counting to 24. I drew four, and he completed the array. He was unsure how to interpret the representation in terms of how many biscuits each person had, but 'saw it' once I added the stick men.

The first time I used the Baking scenario with Nelson, he used cubes as Julie did. I used it in a later session, and this time he elected to try a drawing method, producing this sharing representation.
He began with 24 marks in a linear arrangement, drew vertical lines after each four, and then added the four lower marks. He then 'dealt' the eight marks on the right into the groups, 'transferring' and crossing them out one at a time. From his explanation it seems he estimated they would get four each, then adjusted his model by dividing the remainder – a valid and efficient method often used in adults' mental calculations. Nelson also appeared to find grouping easier, and could solve many with just his fingers.

**Teams**

This was a scenario I improvised for Michael, to explore his difficulties with sharing and possible perceptual difficulties. I asked him to sort out a group of children into two teams, to play tug-of-war. After checking he was familiar with the game, I provided eight cubes and drew a 'rope'. His first answer had cubes only on one side, the left. I asked what would happen when they pulled on the rope, and he said that it would rip. I pointed out that they were all pulling in one direction and there was nobody to pull the other way, and his response was to remove one cube. I suggested that the 'children' left over might like to join in too, and after some thought, he added three to the right side, and then eventually the last cubes, one on each side. Although his final solution was valid, the earlier ones were quite perplexing, and I can make no suggestion to explain their marked asymmetry, other than that it could be a neurocognitive issue.

To test whether a change of layout would produce a change in response, I told
him that the children were now going to play volleyball (checking that he was familiar with the game) and needed to be shared into two teams, provided ten cubes, and drew a 'net'. Michael placed the cubes in pairs, all on the same side.

As before, when I pointed out the problem (in this case, nobody to hit the ball back), he changed his answer, this time moving one pair of cubes across, and then eventually sorted out the others. It is interesting to note that Michael's final (correct) answers on both occasions were presented to me with much greater confidence than his earlier (incorrect) ones. From this I construe that he could recognise a representation which showed objects shared equally, but did not necessarily know how to achieve this with the given materials, other than by trial and error.

**Painting**

This scenario included mathematical concepts already covered, but applied to continuous rather than discrete quantities. The first question described a number of children painting a wall, and each being allocated two metres of it. Terence, Liam and Nelson immediately recognised it as multiplication and answered quickly, the latter two using their fingers. Julie indicated she did not know how to answer it, but understood when I started drawing, as did Michael (although somewhat more slowly).

All participants except Terence used drawings for the *sharing* questions. Liam and Julie needed me to draw a 'wall' of appropriate length for them (despite the fact
that Julie had successfully used a very similar one minutes earlier) but then used it effectively, in Liam’s case extending the representation to encompass *remainders*.

Michael drew his own, and became perceptibly quicker at solving the problems I posed during this session. Nelson, on being asked to share 24 metres of wall between eight children, chose to treat the question in the same way as in *Baking*, i.e. as 24 discrete marks for the wall and eight similar marks for the children, this time using a system of connections (below).

![Diagram of wall division](image)

**Buses**

Another *grouping* scenario, this was intended to introduce the idea of remainders to those who had not encountered them yet, and for those who had, to provoke thought on how one might answer disembedded divisions and practical grouping problems differently.

Again, Terence was able to answer all questions without external representation, but provided evidence of internal visualisation. On being asked how he would have answered if it had not been a question about children and minibuses, he answered "I can still visualise something in my head ... like melons or something". It is highly likely that he picked up the term ‘visualise’ from me during the fieldwork period, but this does not mean his explanation is not valid. Liam also did all questions in his head (with some reluctance), but denied that he "pictured anything in [his] head".

Julie suggested using cubes for these questions, but as she had already proved competent at this, I suggested she try doing them with just pen-and-paper, as
an alternative. This worked equally well, but with this representation she was also able
to deal with remainders. I suggest this difference may be due to the permanence of
marks on a page, as opposed to arrangements of cubes, where it is easy to make any
left-overs 'disappear'. (As this was quite a breakthrough I did not point out that within
the scenario, the answer must be 3 buses.)

\[
29 \div 12 = 2 \text{ r} 5
\]

Nelson used his previous preferred method of representation, but this time with less
success (below). As I had thought might happen, he became confused over which
marks were people and which were buses (understandably, as they looked exactly the
same!) However, as I was now familiar with his method, I could follow and assist.

Nelson was comfortable with his own system, but willing to experiment with an
alternative, and when I suggested it might be easier to show the buses containing the
people, he produced the version below, with which, in fact, he seemed equally
comfortable.
By this stage of the tuition process, Michael had become quite used to the idea of drawing mathematical scenarios. Although he initially said he did not know how to solve the problem, he then suggested drawing a bus, and people on it. This method was very successful, even with numbers which did not divide exactly, although with his level of detail, the process was extremely long-winded for larger numbers (see figure below, and Appendix 5). In fact, in this question I realised I had set Michael a huge challenge, and watched him particularly carefully for signs of distress throughout; however, there seemed to be none (apart from occasional very brief moments of annoyance) and afterwards he was clearly very pleased with his achievements.
Discussion

All participants used visual representation to some degree, and with varying levels of abstraction and complexity. They were also able to comprehend and use representations I created, on some occasions subsequently adopting these and on others rejecting them in favour of their own previous method. Participants had their preferred ways of representing calculations, often with secondary 'back-up' strategies. Representations also depended on question type, Michael and Nelson treating grouping and sharing in different ways, while Terence and Liam used strategies which treated them as interchangeable. Julie also sometimes mixed grouping with sharing, but it was unclear whether she did so because she knew the two would produce the same answer, or had misunderstood the question.

It may appear strange that Terence, selected for his love for drawing, produced the fewest visual representations of all participants. However, this is deceptive. Firstly, although I used considerably larger numbers with him than the others, he still found the majority of the work far easier than expected (from his educational records); however, when working on one of the extension topics which he found more challenging, fractions, he did make considerable use of diagrams (below), while I wrote down the calculations. Secondly, Terence could hold considerably more numbers and operations in his working memory than the others. He informed me how much he disliked writing down his working, but was keen to talk about his mental calculation methods, and gave clear indications of some rich internal visualisation of numbers and operations, even when the questions were posed in an abstract way. (These often revolved around food; for example, the three squares below spontaneously became roast beef sandwiches which had to be shared between four, and during another question he began talking about numbers of bananas.)
At the end of the last tuition session, I asked each of the participants about the visual representations used, and which they preferred using. Michael was unsure whether visual representations had helped him or not, but after some thought, concluded "it was quite good, drawing them". Julie and Liam said that both cubes and drawings were helpful, although Julie said she would use drawings in class, and Liam that he would probably just "use [his] hands". Terence preferred internal visualisations, but would use cubes or drawing if necessary. Nelson was very keen indeed, saying that both types were "very helpful, especially in tests", when he would use a visual representation "to double-check it in [his] own way". Overall, these responses were very positive.
Conclusions

I return to my original research questions.

• What types of visual representation are there in the mathematical experience of the students at this school?

The school uses image-rich textbooks (although there is considerable variation in students' perceptions of how useful these images are) and teachers affirm their belief in the importance of visual methods in teaching/learning. It was not possible in this project to compare different educational environments, but I can comment from previous experience that staff appear to place more emphasis on visual methods than in other schools I have encountered. Students have ready access to concrete materials including cubes, Base-10 blocks and Cuisenaire rods; the degree to which students are encouraged to use these (or to draw problems) varies greatly, and likely depends much on the perceived arithmetical abilities of the student.

While some students actively ask for concrete materials, others do not, even when they would probably help (also applying to drawing). This may be because: the student has not previously found these methods helpful; he/she has found them helpful but it does not occur to him/her to use them unless directed; he/she has not had enough practice to know how to apply them. Cultural factors are also indicated (e.g. Liam using cubes when with me, which, according to his teacher, he does not like doing in class, in front of peers). This last example begs further questions: do students perceive there to be a hierarchy of representation types, and does this affect their choice of method in group situations? Observations suggest this is likely.

As seen from the reproduced images, there was great variety in students' drawings, and even those which used similar 'notations' (e.g. 'tally-mark' representations) used them in structurally different ways. Types of representation and levels of abstraction and/or detail not only varied from student to student, but for the
same student on different question types, or even different occasions. It was considered that there could be a relationship between students' overall mathematical ability and their positioning on the hierarchy of representational types (derived from Bruner); however, this is too simplistic. Dowker (2005, p41) reports that individuals "vary their cognitive style according to the particular type of arithmetical task; the social context; or their level of knowledge and understanding with respect to the problems being presented", and I suggest that this could apply equally well to their representational style. To these factors I add a 'trade-off' situation between speed and reliability; which of these has the stronger effect depending on the situation and the individual's personality (as in LeFevre et al's (2003) work on adults' strategic arithmetic choices, e.g. the 'perfectionist' trait).

- **How can the use of visual representation facilitate students' understanding of mathematical principles and procedures?**

Regarding the principles of multiplication and division, visual representation appeared to be a very important tool for improving students' understanding. For example, for Michael cubes served to clarify the meaning of multiplication and division; for Julie and Nelson, who had an understanding of the meaning, cubes enabled them to 'see' what actually happened in these procedures. Another example is the use of rectangular arrays as a means to discover or 'prove' (in an informal sense) the commutative principle. For students unaware of the principle (Michael and Julie), this representation provided a non-verbally-based comprehensible introduction to the principle and its use, while for those aware that one can reverse the numbers in multiplications, the same representation convinced them that this was more than a handy 'trick' but a true mathematical principle which applies to all cases.
• **How can the use of visual representation aid low-attaining students' mathematical problem-solving?**

Visual representation appeared to be an important problem-solving tool for all students, although varying in importance depending on various factors including how difficult they found the task. For example, in both disembedded and scenario-based questions, both cubes and drawings enabled Michael and Julie to work through problems they were otherwise unable to do; in Nelson's and Liam's cases, to solve problems using larger numbers than they could normally manage; in Terence's case, cubes functioned as a back-up strategy for internal representations, and drawings helped him visualise and so better understand fractions.

**Recommendations**

Inspection of children's representations, particularly so when one can watch the creation process, can illuminate much about a child's thought processes that he/she would be unlikely to put into words. In addition to being of direct help to many children struggling with mathematics, encouraging them to draw or create models of problems, and then observing the process, also has great potential value for teachers, for both assessment and planning.

However, children often do not seem to realise the value of visual representation in mathematics. How has this situation come about? Specialist concrete materials may not always be available, but pencil and paper are. Visual representation is common in the early years of school; when and why does it stop being so? Perhaps children of a certain age are assumed to be able to manage without? If so, this assumption must be challenged. Perhaps it is assumed that children with a 'visual thinking style' will choose to use images, and those who do not use them are not 'visualisers'? Yet the use and potential applications of this skill, as for any tool, need to be taught and practised, both at early stages and later, including applications to more
advanced curriculum material (which may also help reduce the stigma and primary school associations). These are all questions for future research.

I finish with an expressive image, created by another 12-year-old at the school when I asked him to "describe maths".
## Appendices

### A1: Glossary of abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>EBD</td>
<td>Emotional and/or Behavioural Difficulties (sometimes 'Disorder')</td>
</tr>
<tr>
<td>EP</td>
<td>Educational Psychologist</td>
</tr>
<tr>
<td>IEP</td>
<td>Individual Education Plan</td>
</tr>
<tr>
<td>KS3/4</td>
<td>Key Stage 3/4</td>
</tr>
<tr>
<td>LSA</td>
<td>Learning Support Assistant</td>
</tr>
<tr>
<td>SEN</td>
<td>Special Educational Needs</td>
</tr>
<tr>
<td>SENCO</td>
<td>Special Educational Needs Co-ordinator</td>
</tr>
<tr>
<td>SpLD</td>
<td>Specific Learning Difficulties (sometimes 'Disorder')</td>
</tr>
<tr>
<td>WISC</td>
<td>Wechsler Intelligence Scale for Children</td>
</tr>
<tr>
<td>WOND</td>
<td>Wechsler Objective Numeric Dimensions</td>
</tr>
</tbody>
</table>
A2: Questions used in problem-solving interviews

I am interested in how people get on with maths.

1. Tell me how you feel about maths.
   - Are there any bits in particular that you like?
   - Are there any bits in particular that you don't like?
   - Have you always felt like that, or has it changed?
     [- Why do you think it has changed?]

2. Do you find maths easy or hard compared to other subjects?
   - Are there any bits in particular that you find easy?
   - Are there any bits in particular that you find hard?
   - Have you always felt like that, or has it changed?
     [- Why do you think it has changed?]

3. How well do you think that you do in maths, compared to the others in your class?
   - Has it always been like that or has it changed?

I am particularly interested in how people use drawings and diagrams in maths.

4. Tell me about the drawings and diagrams your teacher uses when she is teaching you new things in lessons.
   - Does your LSA also use drawings or diagrams when she/he is helping you in lessons?

5. Can you think of a time when your teacher or LSA has told you to use drawings and diagrams when you are doing Maths work?
   - When they do this, is it usually helpful?

6. Do you ever use drawings and diagrams without being told to?
   - Is it usually helpful?
A3: Mathematical scenarios in problem-solving interviews

The first worksheet, *Stairs*, was taken from Booth & Thomas (2000). It could be answered either in symbolic form through the application of addition and subtraction, in a formalised version through the internal visualisation of movement along a number line, or literally, by drawing the staircase and moving a marker up and down. In addition to changing the numbers and phrasing, I replaced Booth & Thomas’s "baby" with "child", as I considered students might be distracted by thinking about the danger for a baby on a staircase.

The second, *Rose Bushes*, was taken from Booth & Thomas (2000), and also used by Elia & Philippou (2004). It involved a man planting bushes at intervals along a path. It could easily be mistaken for a division question, but actually required an element of reasoning, and almost certainly an internal or external visual representation. In addition to changing the numbers and phrasing, I replaced Booth & Thomas's "trees" with "rose bushes", as they seemed more likely objects to be planted at intervals of one metre.

The third, *Groups*, was taken from Elia & Philippou (2004), deriving originally from Misailidou (2003). It was mainly a division task, and being of the 'grouping' rather than 'sharing' variety, was transmutable to a reverse-multiplication method (requiring either the 5-times-table or step-counting in fives). Numbers were chosen to reflect the size of the participants' own school. I also included an 'extension task', where proportional information only was given.

Worksheets were printed on cream-coloured paper, as suggested in DFES guidance on dyslexia (2004). I always used purple felt-tip, so any markings made by me (while giving help on the problems) were easily identified on the original documents. All students' worksheets which contain visual representations of some kind are reproduced below.
Stairs

A child was at the bottom of a staircase. He climbed up 5 steps, then climbed down 3 steps, then climbed up 6 steps and was at the top.

How many steps were in the staircase? \( \text{\#} \ 8 \)

The child was at the bottom of a different staircase. This one had 12 steps. He climbed up 8 steps, then down 4 steps, then up 5 steps.

How many steps would he need to climb up to get to the top? \( \text{\#} \ 3 \)

How many steps would he need to climb down to get to the bottom? \( \text{\#} \ 9 \)
Rose Bushes

A man had a path in his garden, which was 5 metres long, and straight. He planted a rose bush at each of the two ends, then planted a bush every metre along the path.

How many bushes were planted along the path altogether? 6

The man's neighbour had a path which was 10 metres long, and also straight. He planted a rose bush at each of the two ends, then planted a bush every 2 metres along the path.

How many bushes were planted this time? 6
Groups

A teacher has 20 pupils. She puts them into groups of 5, with 2 girls in each group. How many of her pupils are boys and how many are girls? 12 8

There are 100 pupils in the school, and they are put into groups of 5. This time, there are 5 groups of 5 girls.

How many boys are in the school, and how many groups of boys is this? 75
Stairs

A child was at the bottom of a staircase. He climbed up 5 steps, then climbed down 3 steps, then climbed up 6 steps and was at the top.

How many steps were in the staircase?

The child was at the bottom of a different staircase. This one had 12 steps. He climbed up 8 steps, then down 4 steps, then up 5 steps.

How many steps would he need to climb up to get to the top?

How many steps would he need to climb down to get to the bottom?
Rose Bushes

A man had a path in his garden, which was 5 metres long, and straight. He planted a rose bush at each of the two ends, then planted a bush every metre along the path.

How many bushes were planted along the path altogether?

The man's neighbour had a path which was 10 metres long, and also straight. He planted a rose bush at each of the two ends, then planted a bush every 2 metres along the path.

How many bushes were planted this time?
Groups

A teacher has 20 pupils. She puts them into groups of 5, with 2 girls in each group. How many of her pupils are boys and how many are girls?

8 girls

12 boys

There are 100 pupils in the school, and they are put into groups of 5. This time, there are 5 groups of 5 girls.

How many boys are in the school, and how many groups of boys is this?

75
Rose Bushes

A man had a path in his garden, which was 5 metres long, and straight. He planted a rose bush at each of the two ends, then planted a bush every metre along the path.

How many bushes were planted along the path altogether? □

The man's neighbour had a path which was 10 metres long, and also straight. He planted a rose bush at each of the two ends, then planted a bush every 2 metres along the path.

How many bushes were planted this time? 6

Groups

A teacher has 20 pupils. She puts them into groups of 5, with 2 girls in each group. How many of her pupils are boys and how many are girls? 12 boys 8 girls
Stairs

A child was at the bottom of a staircase. He climbed up 5 steps, then climbed down 3 steps, then climbed up 6 steps and was at the top.

How many steps were in the staircase?

The child was at the bottom of a different staircase. This one had 12 steps. He climbed up 8 steps, then down 4 steps, then up 5 steps.

How many steps would he need to climb up to get to the top?

How many steps would he need to climb down to get to the bottom?
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How many bushes were planted this time?
Groups

A teacher has 20 pupils. She puts them into groups of 5, with 2 girls in each group. How many of her pupils are boys and how many are girls?

There are 100 pupils in the school, and they are put into groups of 5. This time, there are 5 groups of 5 girls.

How many boys are in the school, and how many groups of boys is this?

One day lots of the pupils have colds and only 60 come to school. The headteacher notices that there are twice as many boys as girls.

How many girls are in school that day, and how many boys?
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How many bushes were planted this time?
A4: Two components of the Numeracy Recovery programme

*Word problem solving.* This component involves comprehending addition and subtraction story problems of various semantic types (DeCorte & Verschaffel, 1987); selecting the appropriate operations; and solving the problems.

*Intervention.* Children are given addition and subtraction word problems, which are discussed with them: 'What are the numbers that we have to work with?' 'What do we have to do with the numbers?' 'Do you think that we have to do an adding sum or a taking-away sum?' 'Do you think that John has more sweets or fewer sweets than he used to have?, etc. They are encouraged to use counters to represent the operations in the word problems, as well as writing the sums numerically.

(Dowker, 2005)

I used a similar format to this in my tuition, but with multiplication and division replacing addition and subtraction. I also chose multilink cubes rather than counters, as when put in a rectangular array they may be linked to the 'area' model for multiplication, and rectangles drawn on the (1cm) squared paper. (These changes also apply to the following component.)

*Translation between arithmetical problems presented in concrete, verbal and numerical formats.* ... The intervention programme includes tasks of translating in all possible directions between numerical (written sums); concrete (operations with counters); and verbal (word problem) formats for both addition and subtraction.

For example, in translating from verbal to numerical, children were presented with word problems (e.g. 'Katie had five apples; she ate two, so now she has three left') and asked to 'write down the sum that goes with the story'.

*Intervention.* Children are shown the same problems in different forms and shown that they give the same results. They are also encouraged to represent word problems and concrete problems by numerical sums and to represent numerical problems and word problems by concrete objects.

(ibid.)

As my tuition sessions were flexible in structure, all possible directions were not covered systematically. However, my intention was to use the majority of problem translation formats with each participant, depending on ability. I initially encouraged them to write down the 'sums' for otherwise-represented problems, but if difficulties with the writing process appeared to be distracting them from thinking about the principles involved, I completed part or all of the writing for them as they spoke.
A5: Transcript from tutorial with Michael (Buses scenario)

04/07/2006

CF: Ok, I'm going to give you one more that's a bit harder. How about... see if you can work out, if there were a class of thirty children. [writes 30]

M: Ok, so there's thirty children.

CF: Yes. How can you work out how many buses they'll need?

M: Thirty buses – no! Um...

CF: No, they don't need one each. They can get seven on each bus. [long pause] Could you maybe work it out the way we did before?

M: Erm... so, how many buses will they need?

CF: Mm.

M: Um, thirty people?

CF: Yeh.

M: Ok... [long pause]

CF: I think maybe drawing would help, like it did before.

M: Ok. So I draw a big bus.

CF: Well, the buses are the same size. Room for seven people. [M draws bus] Ok, so seven of them can go on there.

M: Shall I write, shall I draw seven?

CF: I think that would be -

M: Ok, I [?] [starts drawing people on bus – does 2 with head, body, arms]

CF: You don't -

M: I don't think I'm supposed to...

CF: No, that's fine. All I was going to suggest was that you don't have to draw the whole person, unless you want to.

M: [?]

CF: I didn't draw, I just put a blob for a person.

M: Maybe I'm not doing this one...

CF: What you're doing is fine.

M: I'll cross it out. [scribbles out people]

CF: But it was fine, what you've done so far. You can start again if you like, but there's nothing wrong with that.

M: So, I don't get it, though. So I draw seven people, on the bus.

CF: Mm hm. I want to know how many buses we need for thirty people.

M: One, two, three, four... [draws 7 people on bus]

CF: That's right. So that's seven. [writes 7]
M: Ok, [?]. So I draw another?

CF: Yeh, because we've only taken seven pupils so far. But you might not want to
draw it in such detail, because we're going to have a few of these, and you want to
make it easier for yourself.

M: Ok. [draws 2nd bus & 7 people, with same amount of detail]

CF: Ok, so that's another seven. [writes 7]

M: Yeh.

CF: How many is that altogether, so far?

M: Um, one, two. I mean... [counts people] fourteen.

CF: Ok, I'm going to make a note of that up here. [writes 1 ... 7, 2 ... 14] So with one
bus we had seven -

M: Ok.

CF: With two buses, that's fourteen.

M: Ok.

CF: But there's thirty people that need to be taken. So we need more buses, don't
we? Two isn't enough. [M does another bus & people] Ok. Now that was fourteen up to
there...

M: Ok.

CF: Can we count on seven more? From fourteen... How many is that altogether?

M: Ok.

CF: It was fourteen up to here. [indicates]

M: Ok. [counts from start, not from where CF is pointing] Fourteen.

CF: Fourteen.

M: Fifteen... [counts remaining]... twenty-one.

CF: So, that's twenty-one. [writes 3...21]

M: Twenty-one.

CF: Have we got all thirty on the buses yet?

M: Pardon? Have we got what?

CF: Are there thirty people there?

M: Thirty people there, no.

CF: Ok, need another bus.

M: Ok. [draws]

CF: Ok. Up as far as there was twenty-one, yeh?

M: Yeh.

CF: So, we don't need to count them all from the start, do we?

M: No.
CF: Cause we know that that, up to there is twenty-one. So can we carry on counting from there? Twenty-one...

M: Twenty-one... [counts slowly at first, then speeding up] twenty-eight.

CF: Right, so four buses, and that's twenty-eight pupils, twenty-eight children. [writes 4 ... 28] Is that all of them yet? Have we done thirty?

M: Not yet.

CF: Ok, so...

M: We need one more.

CF: We're going to need one more. Remember, there's thirty pupils altogether.

M: So, thirty this time?

CF: We were up to twenty-eight, yeh?

M: Yeh. Need more... [draws another bus]

CF: So the next one is going to be, next person is number...

M: Seven... [goes to start counting from beginning]

CF: Well, rather than count them all up from the start, we counted up to twenty-eight here, yeh?

M: Yeh.

CF: So how many more people are left over?

M: Two. [draws person]

CF: So, that's person twenty-nine...

M: [draws another] Thirty.

CF: And that's person number thirty. Yeh?

M: Good, yeh!

CF: Well done!

M: Thank you.

CF: So, is that all of them, then?

M: Yeh.

CF: There we go, there's our thirty people.

M: [?]

CF: Hm?

M: Should we count it?

CF: We could do, but we counted them as we were going along, didn't we?

M: Ok.

CF: And I checked your counting each time. And that was seven, and then to there was fourteen, and then these made twenty-one, and then these ones made twenty-eight...
M: Yeh.
CF: Twenty-nine, thirty.
M: Ok.
CF: So we needed, how many buses for them altogether?
M: Thirty. I mean, what did-, say it?
CF: How many buses?
M: [counting while pointing] One, two, three, four, five.
CF: Five, that's right. That seemed to help you.
M: Yeah, sort-, a little bit. [?] it started to get harder. Sometimes I thought I didn't get it right.
CF: Mm, no, you were doing it right.
M: Yeah, I know. [looks very pleased]
CF: It's, it can be quite a slow way of doing it, but it was all right, everything you did. It got the right answer.
M: Yeah.
References


