Spatial structuring, enumeration and errors of S.E.N. students working with 3D arrays

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The move from understanding and working with arithmetical structures in one dimension (i.e. additive) to two or more dimensions (i.e. multiplicative) requires a significant change in children's thinking. This paper investigates the varied and developing strategies and understandings of young people struggling with that change, through a series of 3D array enumeration tasks. Participants relied heavily on counting-based strategies, and a new analytical framework is proposed with which to diagnose initial (mis-)conceptions and observe microprogressions on the path towards multiplicative understanding.

Keywords: numeracy, counting, arithmetical strategies, multiplicative thinking, low attainment.

THEORETICAL BACKGROUND

2D and 3D arrays

The 2D rectangular array is a standard visuospatial representation for working with multiplicative structures, such as solving simple multiplication and division problems, or connecting replicating spatial patterns with repeated addition and multiplication. By age 11, all students in UK mainstream education will have encountered rectangular arrays (both dots and grids), and these representations linked to multiplication. The 2D array is thought to have particularly good potential for supporting reasoning in multiplication, and is one of the best for demonstrating the commutative and distributive laws (Harries & Barmby, 2007).

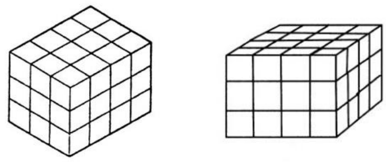
While clearly a powerful tool, the 2D array is limited in terms of enumeration. With a 3D array, the options are more complex: when all dimensions are >2 units, simply counting the visible cubes will not work, as there are non-visible interior cubes; successful enumeration must rely on conceptualising the organisational structure of the array as a space-filling object. While the expected final, formal strategy for students would be a three-dimensional multiplicative formula, on the way to this stage, there are various potential concrete, visuospatial strategies in which the cuboid structure is deconstructed into manageable parts, such as a stack of 2D layers.

An influential series of writings on 3D array tasks (Battista & Clements, 1996, 1998; Battista, 1999) introduced the concept of *spatial structuring,* which I adopt.

> We define spatial structuring as the mental act of constructing an organization or form for an object or set of objects. The process . . . includes establishing units, establishing relationships between units . . . and recognizing that a subset of the objects, if repeated

properly, can generate the whole set (the repeating subset forming a composite unit). (Battista & Clements, 1996, 282)

Ben-Haim's work during the 1980s on 3D arrays involved students interpreting isometric drawings of cuboids (e.g. Figure 1a), so requiring participants to interpret a pattern of identical rhombuses as a solid object. Thus, his set of proposed error types reflects students' tendency to interact with the presented image as a flat object ("1. counting the actual number of faces showing" (Ben-Haim, Lappan, & Houang, 1985), or to have difficulty picturing the cubes not shown. During the 1990s Battista's research on 3D arrays used line drawings with perspective projection (e.g. Figure 1b). His expansion of the set of error categories (Battista & Clements, 1996) reflects similar difficulties. Thus, there is impetus to observe the strategies and errors when participants are, instead, simply presented with the solid shape itself.



Figures 1a-b: Isometric and perspective cuboid images

Counting in multiplicative tasks

The use of counting 'in ones' as a major strategy in additive and multiplicative situations is generally associated with younger children, but persists through adolescence and indeed, adulthood as a supplementary or back-up strategy. However, there is a distinction to be made between retaining counting as a backup, and relying on it as a primary enumeration strategy. Studies (e.g. Siegler, 1988; Geary, Bow-Thomas, & Yao, 1992; Gray & Tall, 1994) indicate that children with arithmetical difficulties are more likely than their typically-attaining peers to rely on counting-based strategies (compared with, e.g. retrieval or derived fact strategies).

Anghileri (1995, 1997) describes a progression in children's counting in multiplicative scenarios, beginning with 'unitary counting', through 'rhythmic counting', to 'skip counting' (or 'step counting'). I propose a refinement: that 'rhythmic counting' is actually made up of two sub-stages, (a) grouping of the numbers and (b) regular rhythmic emphasis of vocalisation or gesture. I use 'grouped counting' for the former, reserving 'rhythmic counting' to describe the specific phenomenon of the musical 'drive' that results from temporally equally-spaced sounds/movements and emphases.

METHOD

The data derive from tasks set during a series of individual or paired problem-solving interviews (with the author), which took place as part of a larger project using microgenetic methods to study emerging and developing multiplicative structure in low-attaining students' use of visuospatial representations. The thirteen participants were aged 11-15, attending mainstream schools in inner London, and identified by their mathematics teachers as numerically weak (compared with peers). All names have been changed.

In each interview, students were presented with a cuboid block formed of multilink cubes, and informed that the blocks were solid, not hollow. The blocks were:

(1) One 3×4×5 cuboid (colours mixed randomly);

(2) One 3×3×5 cuboid (as above);

(3) Two 2×3×6 cuboids, one constructed in three differently-coloured 2×6 layers, the other in six differently-coloured 2×3 layers; students were given the choice which of the two to enumerate;

(4) Two identical 2×2×3 cuboids, both coloured in 2×3 layers; students were asked for the total number of cubes.

The intention was that through these tasks students should come to perceive and use the multiplicative structures inherent in the objects, and their initial and changing conceptualisations could be diagnosed. Specifics of each block were based on the previous observed performances. No time constraint was imposed (actual time varied from 1-15 minutes). If necessary, I provided a series of minimal prompts (described below). Students were allowed to handle the blocks but not pull them apart.

Documentation was via audio recording, photographs, scans of students' papers and observation notes made during and immediately following each interview.

FINDINGS

Task 1: Initial responses

When presented with the first block, all students used counting-based strategies, and all 13 gave incorrect answers. Battista and Clements's analytical categories, while intended for drawn array images, include descriptors also applicable to solids (e.g. "counts outside cubes on all six faces" (1996, 263). However, my students not only made errors in which cubes to count, but in the counting process. Thus one must distinguish an erroneous strategy from errors made in carrying out a correct strategy.

Two students independently made perceptive, effective use of one deconstruction of the array structure, and would have been successful had they not made minor counting errors delivering answers of 59 and 61 rather than 60. With the block on the table, they placed a finger on one of the cubes in the top (4×5) layer and said "1, 2, 3", referring to the touched cube and the two vertically beneath it, then moved the finger along, continuing to group-count threes for every cube in the top layer of 20. It is notable that neither appeared to recognise the set of multiples of three.

Ten of the remaining students began by counting the top layer, then moved onto the other faces of the block, turning it around and attempting to count all the external cubes. Although some students asked for confirmation that the shape was solid as opposed to hollow, their face-based counting strategies nevertheless ignored non-visible interior cubes. Meanwhile, the lack of clear points at which to start and stop counting, and of an obvious 'route' around the six faces, also led to some cubes and/or whole faces being counted more than once, while others were missed. Close observation of gestures and comments indicated that four of the students were attempting to avoid double- or triple-counting cubes, but the other six gave no sign of noticing. The last student unsuccessfully tried to count cubes one colour at a time.

Prompts

If a student was consistently trying to enumerate the squares making up the surface area rather than the cubes making up the volume, I used two prompts: (a) picking up a single loose cube, reminding that these were the items to count; and (b) pointing to a vertex cube, demonstrating how it might be double- or triple-counted. After one or both of these, all students were observed attempting not to over-count edge and vertex cubes. Further prompts drew attention to the layered structure, i.e. the vertical replication aspect of the cuboid shape. The 'layers' prompts were:

Enquiring how many cubes made up the top layer;

Enquiring how many were in the layer underneath (and, if necessary, the next layer);

Commenting explicitly that all layers contained the same numbers of cubes;

Stating the numbers in each layer in the form of an addition (and, if necessary, supporting or performing that calculation).

Six students responded to one of the first three prompts by stating the number of cubes in each layer and calculating a total of 60, while others heard a full demonstration and gave verbal indications that they understood. One (Paula) gave no such indication that she understood either the addition procedure or its relevance.

One student's response indicates the potential effectiveness of a single prompt.

CF: How many are in just the top layer?

Leo: *[Step-counts 5, 10, 15, 20]* Ah!

CF: Does that help at all with getting the total number?

Leo: Well now I think I have a solution to this! If you were able to split this, if you chop the layers off, it'll be 20 there – underneath that is another 20, and underneath that is another 20! *[draws Figure 2]* That's 20 there and 20 there. You could just pull it out like a drawer, then pull that out like a drawer. It would be 20, 20, 20.

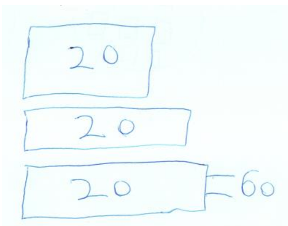


Figure 2: 'Drawers'

Task 2

The second interview, I presented a slightly smaller/easier 3×3×5 cuboid. Two students used a full layers strategy correctly, and three others (including Leo) began by stating the number in the top layer, but then needed one or more prompts. Of the two who had used the columns strategy the first time, one re-used it, while the other tried horizontal rows instead. Four students reverted to counting around the faces, but switched to layers when prompted. Paula again gave no sign of understanding.

Task 3

So far only four students could carry out effective strategies without prompts; one appeared not to follow even complete demonstrations, and all others were at a stage of partial understanding and operationalisation. Hence, in the third interview I highlighted the physical structure by constructing blocks with each layer a different colour. Rather than force students into a single colour structure (and thus numerical structure), I gave them the choice of two equal-sized blocks: a 3-colour block in horizontal 2×6 layers or a 6-colour block in vertical 2×3 layers. This time 11 students used the layers structure; of these, nine were independently successful, while two required prompts. Only two students' initial response was still face-based, and Paula was for the first time able to comprehend and work through the task (with support). The coloured layers were also indicated to be helpful by student comments, such as "[if there are] 6 cubes there [i.e. in one layer] then you know there's six in the rest".

Task 4

The final interview introduced an additional structural aspect: I presented students with two identical colour-layered blocks, and asked for the total number. With a numerical structure of 2(2×3×3), this task extension allowed increased calculation possibilities for the more confident students. Ten produced a correct answer without any arithmetical or strategic support, and the other three succeeded with prompts. All made clear use of the cuboid structure – in particular referring to the coloured layers.

Regarding the duplicate blocks, five students used some form of layer-organised counting for the first, then continued the count similarly for the second. Two students pushed the two blocks together to make them a single mass. Five worked out there were 18 cubes in the first block and doubled (or added another 18) for the total; one more thought of doing this, but was unsure the two blocks were really the same, and insisted on counting the second as well.

ANALYSIS

The data are considered under three distinct but connected analytical aspects: *structuring* (i.e. how the physical structure of the blocks is used by students, and the corresponding numerical structures drawn from them); *enumeration* (i.e. how students used the numbers they derived from the physical blocks), and *errors* (what went wrong in their invention, selection and application of enumeration strategies).

Structuring

The classification for spatial structuring is based on that of Battista and Clements (1996), adapting their descriptors to apply to actual physical objects, re-ordering them into a loose hierarchy, and expanding the category structure.

M	**The student conceptualises the set of cubes as a 3D multiplicative structure**	
		Student finds the length, width and height of the block, and multiplies.
L	**The student conceptualises the set of cubes as a stack of 2D layers**	
	1	*Layer multiplication*: Student computes or counts the number of cubes in one layer, counts the number of layers, and multiplies the two.
	2	*Layer addition*: Student computes or counts the number of cubes in one layer and uses addition or step-counting to get total.
	3	*Counting subunits of layers*: Student's counting of cubes is organised by layers, but the student unit-counts or step-counts by a number smaller than the number of cubes in a layer
C	**The student conceptualises the set of cubes as a 2D array of columns**	
	1	*Column multiplication*: Student counts the number of cubes in one column, counts the number of columns, and multiplies the two.
	2	*Column addition*: Student counts the number of cubes in one column and uses addition or step-counting to get total.
	3	*Counting subunits of columns*: Student's counting of cubes is organized by columns, but the student unit-counts or step-counts by a number smaller than the number of cubes in a column.
F	**The student conceptualises the set of cubes in terms of its faces**	
		Student counts one or more faces of the cuboid. They may be counting cubes (partial volume) or counting squares (surface area).
O	**Other**	Student uses a conceptualisation other than those described above.

Table 1: Spatial structurings of a 3D array

Apart from the two C3-strategy students, initial responses to the task lacked awareness of the array structure. Students interacted with one face at a time, failed to coordinate orthogonal views from different perspectives, and in many cases did not even have a complete faces-based conceptualisation (i.e. surface area). All showed increased awareness of structure following prompts, but the amount of prompting required and strategic change observed varied widely. There was a general move from F towards L strategies, as would be expected. Only Paula and one other attempted an F strategy on all occasions, and both could identify and use layers (with colours and prompts) eventually. However, there was no clear trend <u>within</u> conceptualisation types, i.e. from L3 to L2 to L1 (or equivalent).

On finding a successful strategy, some students repeated it, while others tried alternatives. Battista and Clements consider layers strategies an indication of “see[ing] the array as space-filling” and having “completed a global restructuring of the array” (1998, 234), while “Those in transition, whose restructuring was local rather than global, utilized [column-based] strategies . . . They had not yet formed an integrated conception of the whole array” (ibid). It is unclear why a columns-based spatial structuring should be considered any less sophisticated than a layers-based one. The former deconstructs a 3D array into a 2-dimensional array of 1-dimensional stacks, the latter a 1-dimensional stack of 2-dimensional arrays; both are equally valid (and complementary) space-filling conceptions.

Enumeration

The *enumeration* classification is based on that of Anghileri (1997), and may be used in combination with the spatial structuring categories (producing, e.g. C3R).

1	**Multiplication**	
		Student calculates a total without any interim step-counting.
2	**Step-counting/Addition**	
		Student counts in steps formed of the cardinal number of each layer or column, without any interim numbers (i.e. using a number pattern).
3	**Counting**	
	S	Step-counting (within a layer or face)
	R	Rhythmic counting: Student counts each cube individually, in rhythmically consistent sequence, with clear emphases on cardinal numbers of subgroups.
	G	Grouped counting: Student counts each cube individually, but with the count sequence organised into subgroups.
	U	Unitary counting: Student counts each cube individually, with no grouping.

Table 2: Enumeration strategies for a 3D array

All students began with some form of counting-based strategy, and overall these were by far the most popular. Four students clearly used multiplication in Tasks 3-4, and there were other instances where language implied multiplicative thinking. However, between unitary counting and multiplication was observed a spectrum of ad-hoc grouped-, rhythmic-, step-counting, and addition, and mixed methods.

Errors

Under the analytical aspect *error* are proposed the four types below, between which are covered all errors observed in this dataset.

Spatial structuring (SS) Student uses an incomplete or incorrect conceptualisation of the array structure, e.g. double-counting edge cubes, not accounting for interior cubes.
Numeric calculation or retrieval (NC) Student makes an error in calculating or retrieving a number fact while multiplying, adding or step-counting, e.g. "three twelves... 12, 24, 38".
Verbal count sequence (VC) Student makes an error in their counting, e.g. "26, 27, 29, 30".
Visuospatial/kinaesthetic (VK) Student makes an error relating to the physical aspect of counting, e.g. desynchronisation of verbal count and gesture, repeating a layer/column, etc.

Table 2: Types of error in enumerating 3D arrays

SS: Issues of spatial structuring have already been covered. While most students' initial responses to Task 1 involved mis-structuring, there were only nine subsequent SS errors.

NC: On nine occasions, students mis-recalled addition facts and number patterns, or unsuccessfully attempted formal 'vertical' addition notation for the layers; however, the predominant preference of low-attaining students for counting-based strategies meant that recall of arithmetical facts or procedures was not often required.

VC: Students were all confident in their ability to unit-count individual cubes, yet there were examples of missing and repeating a single number, and missing out a decade.

VK: The most common error type, 22 instances were observed. Some appeared related to fine motor skills, e.g. 'jumping over' a cube. Students also skipped rows, layers and faces, lost track of their start point, etc. Spatial structuring affected this error type: end point and rotation issues happen when working on faces, but with a columns or layers conceptualisation, the block can remain immobile throughout.

Issues of classification

The framework above is useful for identifying individual trajectories and group trends. However, there may be issues in identifying strategies used, e.g. with a student who works silently or with minimal gestures, and does not have the verbal skills to explain coherently how answers were obtained. With students who do verbalise their work, there may be inconsistencies between what they report and what is observed. On several occasions, students used language of multiplication (e.g. "it's three twelves"), but employed a counting strategy; they perceived the multiplicative structure yet were unable to carry out the multiplication operation in any other way.

CONCLUSIONS

Students' understanding of multiplicative structures

Contemplating the visuospatial patterns within physical structures can reasonably be expected to increase awareness of the numeric structures embodied within, at both more advanced stages and the most fundamental stages. E.g. the motion required to move a pointing finger to the next row (etc.) causes a pause in the verbal count, naturally grouping the counting sequence and emphasising the last number spoken. Thus even an incorrect faces-based spatial structuring of a 3D array contains enough structure to serve a useful purpose for the very weakest. While one might assume that students' enumeration of arrays stems directly from their spatial structuring, the relationship is bidirectional; enumeration can also guide structuring. E.g., a student better at step-counting long sequences of small steps than adding a short sequence of larger numbers may (sensibly) opt for a C2 strategy, despite perceiving the layers. Students may seize on familiar number patterns; e.g. noticing there were five units in a row, column or stack could make that the salient grouping of the physical/numeric structure. If struggling students have access to more than one potential structuring, they can choose the one that best suits their capabilities and preferences.

Development of strategies over time and in response to prompts

On finding their initial solutions incorrect, one might expect the kind of cognitive conflict which results in reflection and adaptation; this did not happen. Some students immediately started to re-count in the same way as previously, i.e. they believed in the efficacy of their strategy, but mistrusted their ability to have carried it out properly. Some acquiesced to failure, while others were engaged enough to argue and insist their answer was correct. However, none independently responded by thinking critically about the strategy they had used and improving it or attempting an alternative. Strategic progression in every case required external input. I suggest individuals' willingness (or otherwise) to try alternatives is linked to their relationship with mathematics (or school); on finding a successful strategy, arithmetically insecure students cling to it, while security allows for experimentation.

Further reflections

This simple task proved extremely rich in information about the nature of individuals' current multiplicative thinking, the 'gaps' in their multiplicative understanding, and the variety of enumerative strategies in use. Presented with a task that could be solved by counting, but where that counting was non-trivial and non-routine, adolescents had to reconsider this most basic of numerical skills, and how to apply it. Use of task variants with the same students on four occasions allowed tracking of their progression in terms of spatial structuring, enumeration and error patterns.

Although the layered spatial structure of a cuboid seems obvious to a teacher, and indeed, seemed obvious to some students once given a minimal prompt, others struggled significantly to conceptualise the array as a coordinated, space-filling structure. The use of minimal, sequential prompts, along with the introduction of colour-defined structure, demonstrated the variation in how much input and effort it can take for a student to 'see it'. Furthermore, individual students took their own paths from an essentially 2D, faces-based conceptualisation to a coordinated space-filling structure, with paths through layers, columns, rows, stacks, and combinations of these. While the ability to perceive multiple structurings is unnecessary in the short term (i.e. for solution of this particular task), I assert that in the wider scheme, it is mathematically advantageous and to be encouraged.

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